A Fourth Order Formulation of DDM for Crack Analysis in Brittle Solids

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Abstract

A fourth order formulation of the displacement discontinuity method (DDM) is proposed for the crack analysis of brittle solids such as rocks, glasses, concretes and ceramics. A fourth order boundary collocation scheme is used for the discretization of each boundary element (the source element). In this approach, the source boundary element is divided into five sub-elements each recognized by a central node where the displacement discontinuity components are to be numerically evaluated. Three different formulating procedures are presented and their corresponding discretization schemes are discussed. A new discretization scheme is also proposed to use the fourth order formulation for the special crack tip elements which may be used to increase the accuracy of the stress and displacement fields near the crack ends. Therefore, these new crack tips discretizing schemes are also improved by using the proposed fourth order displacement discontinuity formulation and the corresponding shape functions for a bunch of five special crack tip elements. Some example problems in brittle fracture mechanics are solved for estimating the Mode I and Mode II stress intensity factors near the crack ends. These semi-analytical results are compared to those cited in the fracture mechanics literature whereby the high accuracy of the fourth order DDM formulation is demonstrated.

1. INTRODUCTION

A variety of analytical, semi-analytical and numerical analyses of cracks have been accomplished by several researchers in the field of fracture mechanics [1–10]. The analytical methods give exact solutions but are usually limited to the crack problems with simple geometry and loading conditions. The approximate methods are either semi-analytic or numeric. The semi-analytical or semi-numerical methods usually give more accurate results. This is due to existence of analytical solutions for the problem by exact methods [11]. The approximate solution procedures are based on the variational methods such as minimum potential energy and Ritz methods [12–15]. Semi-analytical procedures such as the indirect boundary element method are based on the semi-analytical solution of a system of integral equations and/or a system of partial differential integral equations on the boundary of physical problems [16–18]. The well-known numerical procedures used in the recent years includes the finite difference method (FDM), the finite element method (FEM) [19–21], the direct boundary element method (BEM) or the boundary integral method (BIM) [22–25], and the dual boundary element method (DBEM) [25–29]. These methods can be used to solve various complex problems in elasticity and fracture mechanics [30–34].

Recently, the boundary collocation schemes have been used to formulate semi-analytical methods such as the displacement discontinuity method (DDM) which is a subdivision of the dual indirect boundary element method (DIBEM) [35–38].

The higher order displacement discontinuity elements and higher order special crack tip elements have been used to solve some fracture mechanics problems and to increase the accuracy of the first and second mode stress intensity factors which are important in the study of rock fracture mechanics [39–42]. The method was further developed to solve the kinked and curved crack problems [43,44]. In all of the previous works, a system of partial differential integral equations was solved on the boundary of any boundary value problem (BVP) occuring in the field of rock fracture mechanics. The method was also extended to solve the infinite, finite and semi-infinite problems in elasticity and fracture mechanics [40,45].

In the present study, a hybridized semi-analytical method is proposed which incorporates the higher order indirect boundary element method for the crack analysis of the finite and infinite plane elasticity problems. The proposed method is verified against some well-known fracture problems.

Also this method is compared with a third order formulation which showed the higher accuracy of the proposed method while using less elements. Verification and comparison showed the validity and applicability of the method for both finite and infinite problems.

2. DEVELOPMENT OF THE FOURTH ORDER DISPLACEMENT DISCONTINUITY FORMULATION

The displacement discontinuity method is an indirect dual boundary element method developed by Crouch [46] for solving two-dimensional (plane strain) elasto-static problems in solid mechanics with implications in rock mechanics and geological engineering [16]. Displacement discontinuity components in a two-dimensional Cartesian coordinates are defined as the difference between shear and normal displacements on the negative and positive sides of a line crack, respectively (Fig. 1). A general displacement discontinuity distribution $\hat{u}(\varepsilon)$ along a crack length *2a*, is shown in Fig.1 (a). Taking the u_x and u_y components of $\hat{u}(\varepsilon)$ to be constant and equal to D_x and D_y respectively, in the interval (*-a, +a*), Fig. 1 (b) shows two displacement discontinuity element surfaces; one on the positive side of *y* ($y=0^+$) and another one on the negative side ($y= 0^-$). It may be assumed that the displacement undergoes a constant change in value when passing from one side of the displacement discontinuity element to the other side. Therefore, the constant element displacement discontinuities D_x and D_y can be formulated as:

$$D_x = u_x(x,0_-) - u_x(x,0_+), D_y = u_y(x,0_-) - u_y(x,0_+) \quad (1)$$

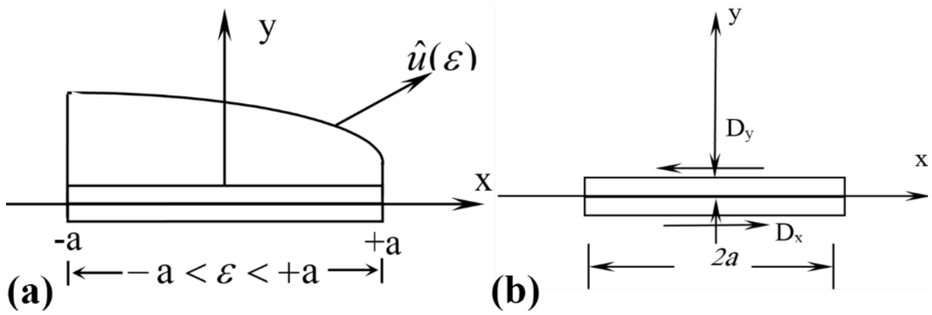

Figure 1. a) Displacement discontinuity distribution, $\hat{u}(\varepsilon)$ along a general boundary element of length 2a; b) The constant shear and normal displacement discontinuity components D_x and D_y with a positive sign convention.

The displacements and stresses for a line crack in an infinite body along the *x-axis* can be written in terms of the harmonic functions *g(x,y)* and *f(x,y)* in the displacement discontinuity version of the indirect boundary element method [16].

Considering a plain strain condition, the shear and normal displacements can be expressed as:

$$\begin{aligned} u_x &= [2(1-\nu) f_{,y} - y f_{,xx}] + [-(1-2\nu) g_{,x} - y g_{,xy}] \\ u_y &= [(1-2\nu) f_{,x} - y f_{,xy}] + [2(1-\nu) g_{,y} - y g_{,yy}] \end{aligned} \quad (2)$$

and, the two-dimensional stress components are:

$$\begin{aligned} \sigma_{xx} &= 2G[2 f_{,xy} + y f_{,xyy}] + 2G[g_{,yy} + y g_{,yyy}] \\ \sigma_{yy} &= 2G[-y f_{,xyy}] + 2G[g_{,yy} - y g_{,yyy}] \\ \sigma_{xy} &= 2G[2 f_{yy} + y f_{,yyy}] + 2G[-y g_{,xyy}] \end{aligned} \quad (3)$$

where G is shear modulus, ν is the Poisson's ratio.

In the fourth order formulation of the displacement discontinuities along an ordinary higher order boundary element of length *2a*, each boundary element is divided into five sub-elements of lengths $2a_1$, $2a_2$, $2a_3$, $2a_4$ and $2a_5$, having the nodal displacement discontinuities D_i^1, D_i^2, D_i^3, D_i^4 and D_i^5 at the five nodes located at the center of the corresponding sub-elements, respectively (Fig. 2).

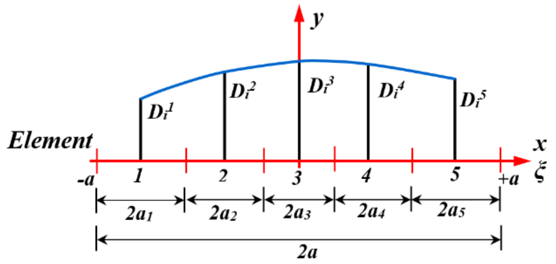

Figure 2. General discretization scheme of a fourth order displacement discontinuity element.

Generally, the displacement discontinuity function, ($D_k(\xi)$) can be divided into five equal sub-elements each containing a central node for which the nodal *DD* is evaluated numerically (the opening displacement discontinuity D_y and sliding displacement discontinuity D_x) [41].

$$D_k(\xi) = \sum_{i=1}^{5} N_i(\xi) D_k^i, \quad k = x,y \qquad (4)$$

Where D_k^1 (i.e. D_x^1 and D_y^1), D_k^2 (i.e. D_x^2 and D_y^2), D_k^3 (i.e. D_x^3 and D_y^3), D_k^1 (i.e. D_x^4 and D_y^4) and D_k^5 (i.e. D_x^5 and D_y^5) are the fourth order nodal displacement discontinuities. A general fourth order displacement discontinuity element can be discretized as shown in Fig. 2.

The displacements and stresses for a line crack in an infinite body along the *x-axis,* in terms of single harmonic functions *g(x,y)* and *f(x,y),* are given in Eqs. 2 and 3 [46], in which these potential functions for the fourth order DD element case can be found from:

$$f(x,y) = \frac{-1}{4\pi(1-\nu)} \sum_{j=1}^{5} D_x^j F_j(I_j)$$
$$g(x,y) = \frac{-1}{4\pi(1-\nu)} \sum_{j=1}^{5} D_y^j F_j(I_j) \qquad (5)$$

in which the common function F_j, is defined as

$$F_j(I_1, I_2, I_3, I_4, I_5) = \int N_j(\xi) \ln \sqrt{(x-\xi)^2 + y^2}\, d\xi, \quad j = 1, \text{to } 5 \qquad (6)$$

where the integrals I_1 to I_5 can be deduced from the shape functions $N_j(\xi)$. The shape functions $N_j(\xi)$ can be obtained from three formulations: i) General sub-elements; ii) Symmetrical sub-elements; and iii) Iso-sub-elements or sub-elements of equal lengths.

Three types of these shape functions may be obtained based on three discretization schemes: (i) Discretization scheme for boundary elements with unequal sub-elements; (ii) Discretization scheme for boundary elements with symmetrical sub-elements; and (iii) Discretization scheme for boundary elements with five equal sub-elements.

2.1. General shape function formulation (unequal sub-elements)

Consider the discretization scheme shown in Fig. 2 in which the third nodal displacement discontinuity D_i^3 is located at the origin of the local coordinate of the boundary element (at the center of third sub-element). The first and second nodal displacement discontinuities D_i^1 and D_i^2 are located on the left side and the fourth and fifth nodal displacement discontinuities D_i^4 and D_i^5 are located on the right side of the central (or third) sub-element, respectively.

The general sub-elements of fourth order (considering $a_1 \neq a_2 \neq a_3 \neq a_4 \neq a_5$ in Fig. 2) in which *a* is crack half-length and $a = a_1 + a_2 + a_3 + a_4 + a_5$ can be defined based on the displacement discontinuity function $D_i(\xi)$ as

$$D_i(\xi) = A + B\xi + C\xi^2 + D\xi^3 + E\xi^4, \quad i = x, y \qquad (7)$$

This formulation can be expressed in form of the shape function $N_j(\xi)$ as

$$D_i(\xi) = \sum_{j=1}^{5} N_j(\xi) D_i^j, \quad i = x, y \qquad (8)$$

By taking the center of the displacement discontinuity element as the origin of the local *x,y* coordinate, the constant $A = D_i^3$ and the shape functions $N_j(\xi)$ can be expressed as:

$$N_j(\xi) = B_j\xi + C_j\xi^2 + D_j\xi^3 + E_j\xi^4, \quad j = 1 \text{ to } 5 \qquad (9)$$

where the constants $B_1, B_2, \ldots, C_1, C_2, \ldots$, etc. are given in appendix A. This formulation is somewhat complicated; therefore, the symmetrical sub-element formulation and the equal length sub-element formulation can be effectively used. However, in the case of crack tips, the general fourth order DD formulation can be effectively used for the crack tips.

The discretization scheme presented in Fig. 3 may be more useful, where $l_1=2a_1$, $l_2=2a_2$, $l_3=2a_3=$, $l_4=2a_4$, $l_5=2a_5$ and $L=2(a_1+a_2+a_3+a_4+a_5)=2a$. For $a_1=l/20$, then $a_2=3l/40$, $a_3=l/10$, $a_4=l/8$, $a_5=3l/20$.

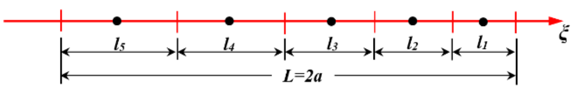

Figure 3. Discretization for a 4th order DD element with sub-elements of unequal lengths.

The fourth order shape functions $N_j(\varepsilon)$ can be defined to write the fourth order variation of the shear and normal displacement discontinuities,

$$D_i(\varepsilon) = [N_1(\varepsilon)]D_i^1 + [N_2(\varepsilon)]D_i^2 + [N_3(\varepsilon)]D_i^3 + [N_4(\varepsilon)]D_i^4 + [N_5(\varepsilon)]D_i^5, \quad i = x, y \tag{10}$$

And *N1(ε), N2(ε), N3(ε), N4(ε)* and *N5(ε)* are the displacement collection shape functions for the corresponding displacement discontinuities, D_i^1, D_i^2, D_i^3, D_i^4 and D_i^5 respectively. Inserting Eq. (7) into Eq. (4) and rearranging the terms, a common special function $F(x,y)$ as given in Equation (6) can be obtained as:

$$F(x,y) = \frac{-1}{4\pi(1-\nu)}\left\{\left[\int_{-a}^{a} N_1(\varepsilon)\ln\left[(x-\varepsilon)^2+y^2\right]^{\frac{1}{2}} d\varepsilon\right]D_i^1 + \left[\int_{-a}^{a} N_2(\varepsilon)\ln\left[(x-\varepsilon)^2+y^2\right]^{\frac{1}{2}} d\varepsilon\right]D_i^2 + \left[\int_{-a}^{a} N_3(\varepsilon)\ln\left[(x-\varepsilon)^2+y^2\right]^{\frac{1}{2}} d\varepsilon\right]D_i^3 + \left[\int_{-a}^{a} N_4(\varepsilon)\ln\left[(x-\varepsilon)^2+y^2\right]^{\frac{1}{2}} d\varepsilon\right]D_i^4 + \left[\int_{-a}^{a} N_5(\varepsilon)\ln\left[(x-\varepsilon)^2+y^2\right]^{\frac{1}{2}} d\varepsilon\right]D_i^5\right\} \tag{11}$$

where the common function $F_j(I_1, I_2, I_3, I_4, I_5)$ given in Eq. (6) can be calculated from the integrals I_1, I_2, I_3, I_4, I_5 as

$$I_1(x,y) = \int_{-a}^{a} \ln\sqrt{(x-\xi)^2+y^2}\,d\xi = y(\theta_1-\theta_2) - (x-a)\ln(r_1) + (x+a)\ln(r_2) - 2a$$

$$I_2(x,y) = \int_{-a}^{a} \xi\ln\sqrt{(x-\xi)^2+y^2}\,d\xi = xy(\theta_1-\theta_2) + 0.5\left(y^2-x^2+a^2\right)\ln\frac{r_1}{r_2} - ax$$

$$I_3(x,y) = \int_{-a}^{a} \xi^2\ln\sqrt{(x-\xi)^2+y^2}\,d\xi = \frac{y}{3}\left(3x^2-y^2\right)(\theta_1-\theta_2) + \frac{1}{3}\left(3xy^2-x^3+a^3\right)\ln(r_1) - \frac{1}{3}\left(3xy^2-x^3-a^3\right)\ln(r_2) - \frac{2a}{3}\left(x^2-y^2+\frac{a^3}{3}\right) \tag{12}$$

$$I_4(x,y) = \int_{-a}^{a} \xi^3\ln\sqrt{(x-\xi)^2+y^2}\,d\xi = -xy\left(x^2-y^2\right)(\theta_1-\theta_2) + 0.25\left(3x^4-6x^2y^2+8a^2x^2+a^4-y^4\right)[\ln(r_1)-\ln(r_2)] - 2ax\left(x^2+a^2\right)[\ln(r_1)+\ln(r_2)] + 1.5ax^3 - 3axy^2 + 7a^3x/6$$

and *θ1, θ2, r1,* and *r2* are defined as:

$$\theta_1 = tg^{-1}\left(\frac{y}{x-a}\right), \qquad \theta_2 = tg^{-1}\left(\frac{y}{x+a}\right),$$
$$r_1 = \sqrt{(x-a)^2+y^2}, and \; r_2 = \sqrt{(x+a)^2+y^2} \tag{13}$$

For an element of length *L=2a*, where *L=l1+l2+l3+l4+l5* (as shown in Fig. 3), the integral, $I_5(x,y) = \int_{-a}^{a} \xi^4\ln\sqrt{(x-\xi)^2+y^2}\,d\xi$, indicating the fourth order displacement discontinuity formulation can be evaluated as

$$I_5 = \int_{x-a}^{x+a} (x-z)^4\ln\sqrt{z^2+y^2}\,dz = x^4B_{11} - 4x^3B_{12} + 6x^2B_{21} - 4xB_{22} + B_{23} \tag{14}$$

By taking *z=x-ξ* and *dξ=-dz* then *–a<ξ<a and x-a<z<x+a*.

It should be noted that the integrals I_1, I_2, I_3 and I_4 and their derivatives are completely explained by the second author in [39]. The shape functions $N_j(\varepsilon)$ and the integral I_5 and its derivatives can be obtained as explained in the appendices A and B of this paper.

2.2. Symmetrical sub-elements formulation

The symmetrical sub-element formulation for the fourth order displacement discontinuity variation along a boundary of length $2a'$ may be performed based on the discretization shown in Fig. 4. The displacement discontinuity function $D_i'(\xi)$ is considered as

$$D_i'(\xi) = A' + B'\xi + C'\xi^2 + D'\xi^3 + E'\xi^4 , \quad i = x, y, -a' < \xi < a' \tag{15}$$

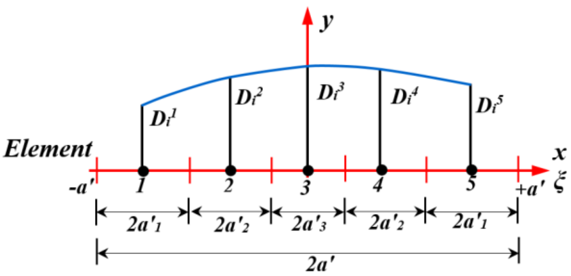

Figure 4. Fourth order collocation for the symmetrical displacement discontinuity elements.

This formulation can be written in the form of shape function $N'_j(\xi)$ as:

$$D_i'(\xi) = \sum_{j=1}^{5} N'_j(\xi) D_i^j \qquad , i = x, y \tag{16}$$

The shape functions $N'_j(\xi)$ for a symmetrical element can be defined as:

$$\begin{aligned} N'_1(\xi) &= B'_1\xi + C'_1\xi^2 + D'_1\xi^3 + E'_1\xi^4 \\ N'_2(\xi) &= -B'_2\xi + C'_2\xi^2 + D'_2\xi^3 + E'_2\xi^4 \\ N'_3(\xi) &= C'_3\xi^2 + E'_3\xi^4 \\ N'_4(\xi) &= B'_2\xi + C'_2\xi^2 - D'_2\xi^3 + E'_2\xi^4 \\ N'_5(\xi) &= -B'_1\xi + C'_1\xi^2 + D'_1\xi^3 + E'_1\xi^4 \end{aligned} \tag{17}$$

where the constants B'_1, B'_2, ..., C'_1, C'_2, ..., etc. are given in appendix A.

2.3. Equal sub-elements formulation

The equal sub-element formulation for the fourth order displacement discontinuity variation along a boundary of length $2a''$ may be expressed based on the discretization shown in Fig. 5 by taking $a'' = 5a''_1$ and $a''_1 = a''_2 = a''_3 = a''_4 = a''_5$. Hence, the displacement discontinuity function $D_i'(\xi)$ may be defined as

$$D''_i(\xi) = A'' + B''\xi + C''\xi^2 + D''\xi^3 + E''\xi^4 , \quad i = x, y, -a'' < \xi < a'' \tag{18}$$

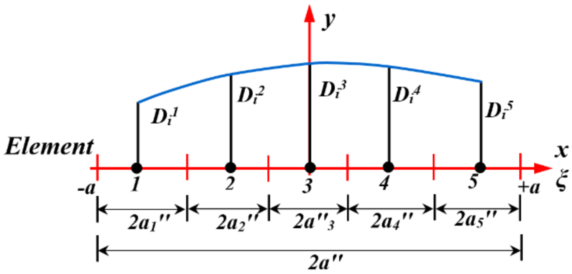

Figure 5. Fourth order collocation for the displacement discontinuities with equal sub-elements.

This formulation can be written in form of the shape function $N''_j(\xi)$ as:

$$D''_i(\xi) = \sum_{j=1}^{5} N''_j(\xi) D_i^j \qquad , i = x, y \tag{19}$$

The constants A'', B'', C'', D'', and E'' are expressed as:

$$A'' = D_i^3, \quad B'' = \frac{1}{24a''_1}\left(D_i^1 - 8D_i^2 + 8D_i^4 - D_i^5\right)$$

$$C'' = \frac{1}{96a''^2_1}\left(-D_i^1 + 16D_i^2 - 30D_i^3 + 16D_i^4 - D_i^5\right)$$

$$D'' = \frac{1}{96a''^2_1}\left(-D_i^1 - +2D_i^2 - 2D_i^4 + D_i^5\right)$$

$$E'' = \frac{1}{384a^4}\left(D_i^1 - 4D_i^2 + 6D_i^3 - 4D_i^4 + D_i^5\right)$$

The equal sub-element formulation of displacement discontinuity can also be used for crack tip discretization.

2.4. Special crack tip element (fourth order displacement discontinuity formulation)

Since the singularities of the stresses and displacements near the crack ends may reduce their accuracies, special crack tip elements are used to increase the accuracy of the *DD*s near the crack tips [39]. As shown in Fig. 6, the *DD* variation for five nodes can be formulated using a special crack tip element containing five nodes (or having five special crack tip sub-elements as shown in Fig. 3).

$$D_{ic}(\xi) = [N_{C1}(\xi)]D_{ic}^1 + [N_{C2}(\xi)]D_{ic}^2 + [N_{C3}(\xi)]D_{ic}^3 + [N_{C4}(\xi)]D_{ic}^4 + [N_{C5}(\xi)]D_{ic}^5 , \; i = x, y \tag{20}$$

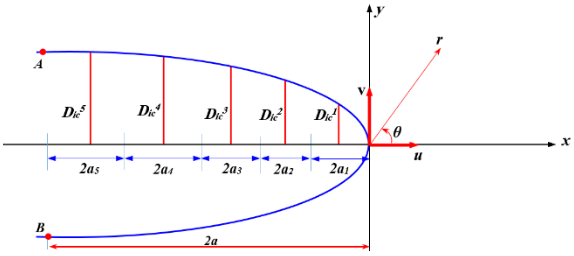

Figure 6. A crack tip with five sub-elements.

Considering a crack tip element with five equal sub-elements ($a_1=a_2=a_3=a_4=a_5$), the shape functions $N_{C1}(\xi)$ to $N_{C5}(\xi)$ can be obtained as equations:

$$\begin{aligned}
N_{C1}(\xi) &= C''_{11}\xi^{\frac{1}{2}} + C''_{21}\xi^{\frac{3}{2}} + C''_{31}\xi^{\frac{5}{2}} + C''_{41}\xi^{\frac{7}{2}} + C''_{51}\xi^{\frac{9}{2}} \\
N_{C2}(\xi) &= C''_{12}\xi^{\frac{1}{2}} + C''_{22}\xi^{\frac{3}{2}} + C''_{32}\xi^{\frac{5}{2}} + C''_{42}\xi^{\frac{7}{2}} + C''_{52}\xi^{\frac{9}{2}} \\
N_{C3}(\xi) &= C''_{13}\xi^{\frac{1}{2}} + C''_{23}\xi^{\frac{3}{2}} + C''_{33}\xi^{\frac{5}{2}} + C''_{43}\xi^{\frac{7}{2}} + C''_{53}\xi^{\frac{9}{2}} \\
N_{C4}(\xi) &= C''_{14}\xi^{\frac{1}{2}} + C''_{24}\xi^{\frac{3}{2}} + C''_{34}\xi^{\frac{5}{2}} + C''_{44}\xi^{\frac{7}{2}} + C''_{54}\xi^{\frac{9}{2}} \\
N_{C5}(\xi) &= C''_{15}\xi^{\frac{1}{2}} + C''_{25}\xi^{\frac{3}{2}} + C''_{35}\xi^{\frac{5}{2}} + C''_{45}\xi^{\frac{7}{2}} + C''_{55}\xi^{\frac{9}{2}}
\end{aligned} \quad (21)$$

Constants C''_{13}, C''_{23}, etc. are given in appendix C.

By substituting Eq. (21) into Eq. (20) and then substituting these equations into Eq. (4) and (5) and following the procedures similar to those given for the derivation of the general potential function $F_j(I_1, I_2, I_3, I_4, I_5)$ in Eq. (6), the general potential function $F_C(x,y)$ for the crack tip element can be expressed as:

$$\begin{aligned}
F_C(x,y) = \frac{-1}{4\pi(1-\nu)} \Bigg\{ &\left[\int_{-a}^{a} N_{C1}(\xi)\ln\left[(x-\xi)^2+y^2\right]^{\frac{1}{2}} d\xi\right] D^1_{ic} + \left[\int_{-a}^{a} N_{C2}(\xi)\ln\left[(x-\xi)^2+y^2\right]^{\frac{1}{2}} d\xi\right] D^2_{ic} + \\
&\left[\int_{-a}^{a} N_{C3}(\xi)\ln\left[(x-\xi)^2+y^2\right]^{\frac{1}{2}} d\xi\right] D^3_{ic} + \left[\int_{-a}^{a} N_{C4}(\xi)\ln\left[(x-\xi)^2+y^2\right]^{\frac{1}{2}} d\xi\right] D^4_{ic} + \left[\int_{-a}^{a} N_{C5}(\xi)\ln\left[(x-\xi)^2+y^2\right]^{\frac{1}{2}} d\xi\right] D^5_{ic} \Bigg\}
\end{aligned} \quad (22)$$

The potential function $F_{Cj}(I_{CK})$ for special crack tip elements can be written in the following form

$$F_C(I_{Cj}) = \int_{-a}^{a} N_{Cj}(\xi)\ln\sqrt{(x-\xi)^2+y^2}\, d\xi, \quad j = 1 \ \text{to} \ 5 \quad (23)$$

And from this, the following integrals are deduced:

$$\begin{aligned}
I_{C1}(x,y) &= \int_{-a}^{a} \xi^{0.5}\ln\sqrt{(x-\xi)^2+y^2}\, d\xi, \quad I_{C2}(x,y) = \int_{-a}^{a} \xi^{1.5}\ln\sqrt{(x-\xi)^2+y^2}\, d\xi \\
I_{C3}(x,y) &= \int_{-a}^{a} \xi^{2.5}\ln\sqrt{(x-\xi)^2+y^2}\, d\xi, \quad I_{C4}(x,y) = \int_{-a}^{a} \xi^{3.5}\ln\sqrt{(x-\xi)^2+y^2}\, d\xi \\
I_{C5}(x,y) &= \int_{-a}^{a} \xi^{4.5}\ln\sqrt{(x-\xi)^2+y^2}\, d\xi
\end{aligned} \quad (24)$$

Following the procedure explained by [39] the complete solution of the integrals given in Eqs. (24) are presented in Appendix D. Based on the linear elastic fracture mechanics (LEFM) principles, the Mode I and Mode II stress intensity factors K_I and K_{II} can be easily deduced [6,39]. A crack tip element of length $2a$ is considered then the stress intensity factors with respect to the normal and shear displacement discontinuities (assuming plane strain condition), are determined [47] as:

$$\begin{aligned}
K_I &= \frac{\mu}{4(1-\nu)}\left(\frac{2\pi}{a}\right)^{\frac{1}{2}} D_y(a), \\
K_{II} &= \frac{\mu}{4(1-\nu)}\left(\frac{2\pi}{a}\right)^{\frac{1}{2}} D_x(a)
\end{aligned} \quad (25)$$

where, μ is the shear modulus and ν is Poisson's ratio of the brittle material.

3. VERIFICATION OF THE FOURTH ORDER DISPLACEMENT DISCONTINUITY FORMULATION

Some example problems such as center slant cracks, curved cracks, and double-edge cracks are being used to verify the fourth order displacement discontinuity formulations presented in this research. The general curved cracks can be considered as elliptical cracks with different ratios of the minor axis (c) to the major axis (b) (i.e. c/b ratio).

3.1. Center slant crack (c/b=0) under far field tension

The center slant crack under far field tension is shown in Fig. 7. The slant angle β = 45° and half crack length a=1m are assumed in the analysis. A crack tip element length to half crack length ratio l/a= 0.1 has been used for the numerical solution of the problem. The analytical solution of the first and second mode SIFs, K_I and K_{II}, for the center slant crack problem are given as [48,49]:

$$K_I = \sigma\sqrt{\pi a}\,\sin^2\beta$$
$$K_{II} = \sigma\sqrt{\pi a}\,\sin\beta\,\cos\beta \tag{26}$$

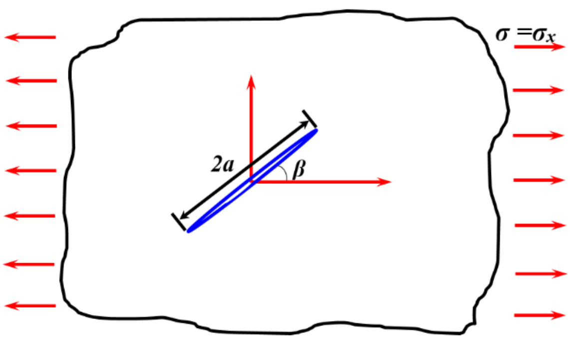

Figure 7. Center slant crack under far field tension.

As it can be seen in Fig. 8 the error in SIF calculation is less than 0.008% for all points except for K_{II} in β=75° which is less that 0.014%. The proposed formulation manages to offer high-precision predictions for all line cracks. In the next section, the accuracy of the formulation in dealing with more complicated cracks, i.e. curved cracks, will be investigated.

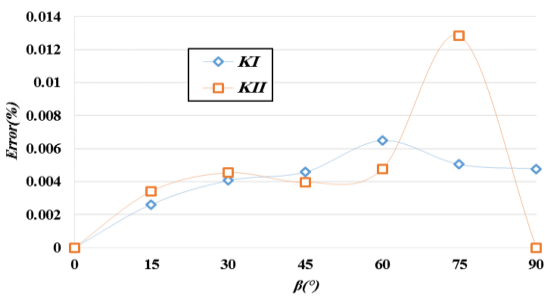

Figure 8. Percentage error in SIFs prediction for central slant cracks using fourth order DDM.

3.2. Circular crack (c/b=1)

Circular (c/b=1) or curved (0<c/b<1) cracks may occur in cracked bodies [39,43,47]. The analytical values of K_I and K_{II} for a general circular crack of Fig. 9 may be expressed as [50]:

$$K_I = \sigma\sqrt{\pi b}\,\frac{\cos(\alpha/4)}{1+\sin^2(\alpha/4)}$$
$$K_{II} = \sigma\sqrt{\pi b}\,\frac{\sin(\alpha/4)}{1+\sin^2(\alpha/4)} \tag{27}$$

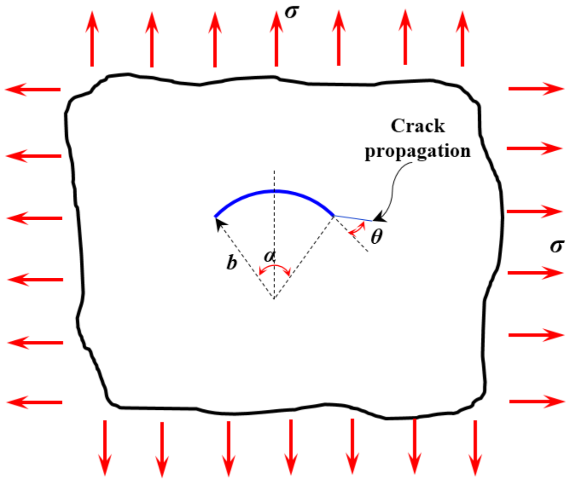

Figure 9. Curved crack in an infinite body under biaxial tension.

Fig. 10 shows the percentage error of numerical values of stress intensity factors, K_I and K_{II}, for different α angles. Fig. 11 shows analytical and numerical values for different α angles. As it can be seen the new formulation produces better results as α grows.

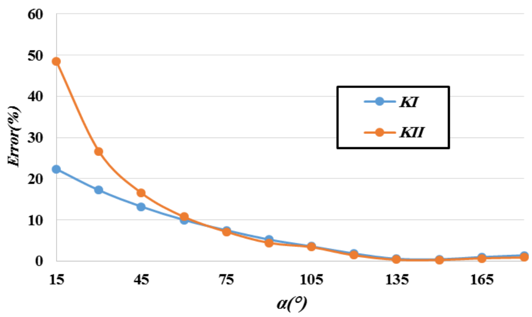

Figure 10. Percentage error in SIFs predictions of various circular cracks.

3.3. Elliptical cracks

The line and circular cracks set two limits for curved cracks defined by the ratio of ellipsis radiuses shown in Fig. 12. Normalized values of SIFs for different curvature ratios from 0 (line crack) to 1 (circular crack) are also presented in Fig. 12.

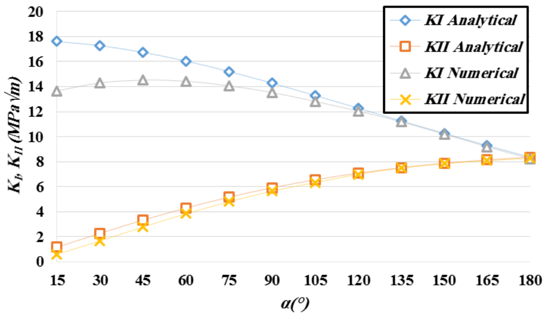

Figure 11. Analytical and numerical values of SIFs for various circular cracks.

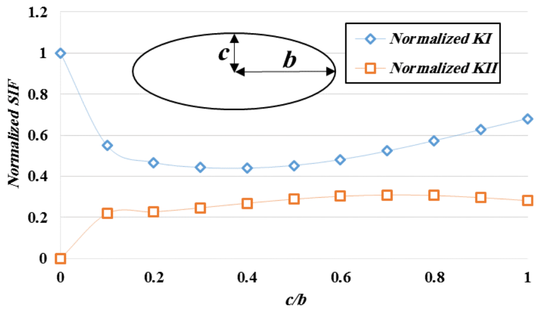

Figure 12. Normalized stress intensity factors for different ellipsis radiuses (c/b ratios) for α=90.

Fig. 13 demonstrates the initiation angle for the same cracks presented in Fig. 12. For c/b=0 (i.e. a line crack) the crack has obviously propagated in its original plane, while for c/b=1 (i.e. circular crack) initiation occurs at about 35° (as predicted from analytical equations).

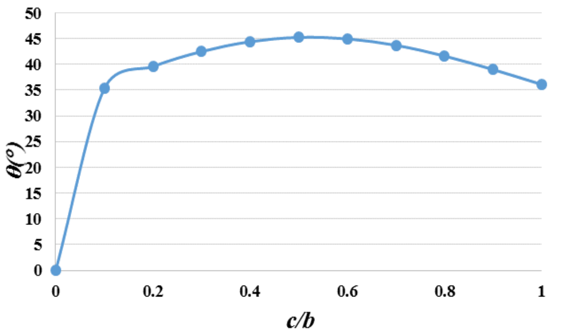

Figure 13. Crack propagation angle for different ellipsis radiuses (c/b ratios) for α=90.

3.4. Double- edge cracks in a finite plane

For the problem shown in Fig. 14, let the width of the plate, *W=1 m*, *b/W* = 0.1, the normal stress, $\sigma = 10\ MPa$ and the Poisson's ratio $\nu = 0.2$, then the analytical normalized value of $K_I/(\sigma\sqrt{\pi b}) = 1.1215$ [6].

The fourth order formulation of displacement discontinuity method is being used to find the normalized values of $K_I/(\sigma\sqrt{\pi b})$ considering the effects of the number of elements along the crack in a double edge crack problem (Fig. 15).

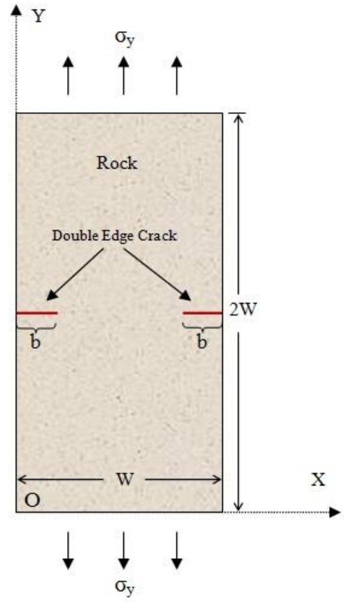

Figure 14. A double edge crack in a finite plate with W=1 m.

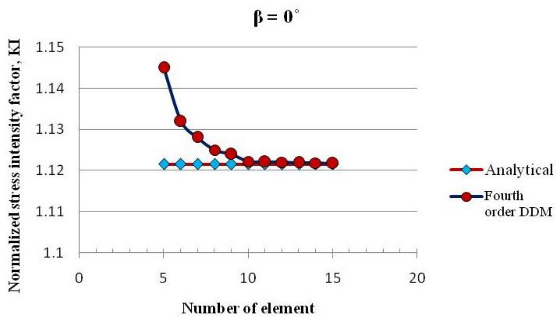

Figure 15. The analytical and numerical values of the normalized Mode I stress intensity factor, $K_I/(\sigma\sqrt{\pi b})$, for the double edge crack, using different number of elements along the crack and l/b=0.1

Let the ratio of crack tip element length, *l*, to crack length, *b*, change as $\frac{l}{b} = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35 \text{ and } 0.4$, and taking the number of elements along the boundaries of the problem as 60 and those along the crack as 10, the computed results for $K_I/(\sigma\sqrt{\pi b})$ are compared with the corresponding analytical value (Fig. 16).

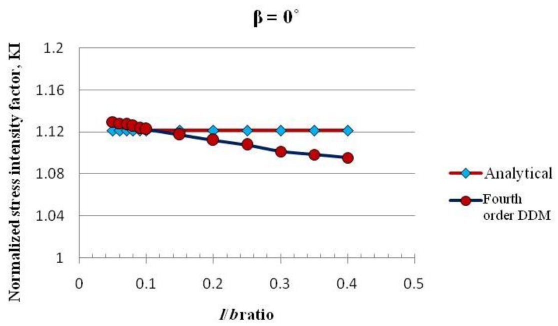

Fig. 16. The analytical and numerical values of the normalized Mode I stress intensity factor, $K_I/(\sigma\sqrt{\pi b})$, for the double edge crack using different crack tip length to crack length ratios (different *l/b* ratios).

Fig. 17 shows the effects of different crack length to width ratios (different *b/w* ratios) on the value of $K_I/(\sigma\sqrt{\pi b})$ for a double edge crack problem. The normalized SIF is evaluated using the third order and the proposed fourth order elements.

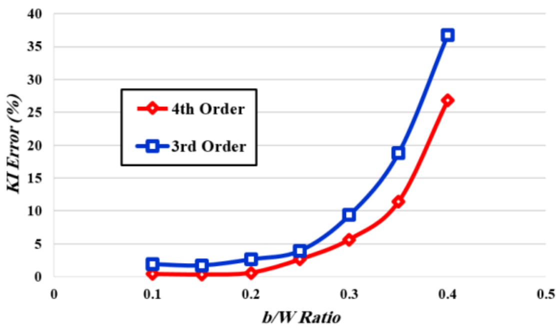

Fig. 17. The error in estimating numerical values of the normalized Mode I stress intensity factor, $K_I/(\sigma\sqrt{\pi b})$, for the double edge crack using different crack length to width ratios (different b/w ratios) for the third and the fourth order elements.

As it can be seen, the proposed higher order element can achieve a higher accuracy in comparison to the previous used third order elements in estimating the stress intensity factor. The error reaches values of around 36% and 26% for the third and the fourth order elements respectively for b/w=0.4. The high values of error are due to boundary effect. Increasing b/w ratio decreases the distance between crack tips and distance of crack tips from the boundaries.

4. CONCLUSIONS

The fourth order formulation of displacement discontinuity for the crack analysis was presented in this paper. Some special discretization schemes were developed to make the formulation more robust and concise. The details of the formulations with different integrals and derivatives required in the fourth order solution of displacement discontinuities along each boundary element were given in the text and relevant appendices. It has been shown that the special crack tip elements can also be formulated based on the fourth order displacement discontinuity formulation presented in this research. Several problems were selected as instances from the fracture mechanics literature to verify the numerical results obtained by the present formulation.

The special singular integrals derived in the solution of stress and displacement fields near the crack tips while using the semi-analytical indirect boundary element method. The new formulations and the corresponding derivatives were given in detail in the text and in the appendices (A, B and C) of this paper. This method is a hybridized semi-analytical method which gives a general solution of the displacement and stress fields near the crack ends based on which Mode I and Mode II (mixed mode) stress intensity factors (i.e. K_I and K_{II} or their normalized forms $K_I/(\sigma\sqrt{\pi b})$, and $K_{II}/(\sigma\sqrt{\pi b})$, respectively), for different example problems of fracture mechanics, can be obtained with a very high accuracy (less than one percent errors in most cases when the higher order elements and special crack tip elements are being used simultaneously). The results obtained by using the 4th order formulations of displacement discontinuity were compared with the corresponding results obtained by using the analytical methods cited in the literature. These comparisons validated the higher accuracy of the present formulation in the crack analysis of brittle solids. The proposed fourth order elements were also compared with the third order elements already available. The comparison showed the higher accuracy of the fourth order elements. The proposed fourth order elements may be used to solve fracture and elastic problems with a higher accuracy.

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APPENDIX A

A.1. Constants for general shape functions:

$$B_1 = \frac{1}{BA_1}\left[1 + C_1(BA_2) + D_1(BA_3) + E_1(BA_4)\right]$$

where BA_1 to BA_4 are:

$$BA_1 = S_2 - S_1\ ,\quad BA_2 = S_1^2 - S_2^2\ ,\quad BA_3 = S_2^3 - S_1^3\ ,\quad BA_4 = S_1^4 - S_2^4$$

and S_1 and S_2:

$$S_1 = a_1 + 2a_2 + a_3,\quad S_2 = a_2 + a_3$$

$$C_1 = \frac{1}{CA_1}\left[CA_2 - D_1(CA_3) + E_1(CA_4)\right]$$

in which:

$$CA_1 = \frac{S_1^2 + S_2^2}{BA_1}, \quad CA_2 = \frac{S_4}{BA_1},$$

$$CA_3 = \frac{BA_3}{BA_1} + S_4^3, \quad CA_4 = -\frac{BA_4}{BA_1} + S_4^4$$

where $S_4=a_3+a_4$

$$D_1 = \frac{1}{DA_1}\left[-DA_2 + E_1(DA_5)\right]$$

in which:

$$DA_1 = \frac{S_2}{BA_1} - \frac{CA_3}{CA_1} S_2 \left[\frac{BA_2}{BA_1} - S_2\right]$$

$$DA_2 = \frac{S_2}{BA_1}\left[1 - \frac{CA_2}{CA_1}\left(BA_2 - S_2(BA_1)\right)\right]$$

$$DA_5 = -S_2\left[S_2^3 + \frac{CA_4}{CA_1}\left(\frac{BA_2}{BA_1} - S_2\right)\right]$$

and E_1 is:

$$E_1 = -\frac{1}{EA_1}\left[EA_2 - \frac{EA_5(DA_2)}{DA_1}\right]$$

$$EA_1 = EA_7 - \frac{DA_3(EA_5)}{DA_1}$$

$$EA_2 = \frac{S_4 - S_5}{BA_1} - CA_2(EA_0)$$

$$EA_6 = EA_0(CA_3) + \left(S_5^3 - S_4^3\right),$$
$$EA_7 = EA_0(CA_4) + \left(S_5^4 - S_4^4\right)$$

where $S_5=a_3+2a_4+a_5$ and EA_0 is:

$$EA_0 = \frac{BA_2(S_4 - S_5)}{(BA_1)(CA_1)} + \frac{S_4^2 - S_5^2}{CA_1}$$

For B_2, C_2, D_2 and E_2:

$$B_2 = \frac{1}{BA_1}\left[C_2(BA_2) + D_2(BA_3) + E_2(BA_4) - 1\right]$$

$$C_2 = \frac{1}{CA_1}\left[CA_2 - D_2(CA_3) - E_2(CA_4)\right]$$

$$D_2 = -\frac{1}{DA_1}\left[1 - DA_4 + E_2(DA_5)\right]$$

where

$$DA_4 = -\frac{S_2}{BA_1}\left[1 - \frac{CA_2}{CA_1}\left(BA_2 - S_2(BA_1)\right)\right]$$

$$E_2 = -\frac{1}{EA_1}\left[EA_3 - \frac{EA_5(DA_3 + 1)}{DA_1}\right]$$

$$EA_3 = \frac{S_4 - S_5}{BA_1} - CA_2(EA_0)$$

For B3 through E3:

$$B_3 = \frac{1}{BA_1}\left[C_3(BA_2) + D_3(BA_3) + E_3(BA_4)\right]$$

$$C_3 = -\frac{1}{CA_1}\left[1 + D_3(CA_3) + E_3(CA_4)\right]$$

$$D_3 = \frac{1}{DA_1}\left[1 - DA_4 + E_3(DA_6)\right]$$

$$DA_6 = -\frac{S_2}{CA_1}\left[\frac{BA_2}{BA_1} - S_2\right]$$

$$E_3 = -\frac{1}{EA_1}\left[EA_4 - \frac{EA_6}{DA_1}\right]$$

$$EA_4 = \frac{BA_2(S_4 - S_5)}{(BA_1)(CA_1)} + \frac{S_4^2 - S_5^2}{CA_1}$$

and for B4 to E4 :

$$B_4 = \frac{1}{BA_1}\left[C_4(BA_2) + D_4(BA_3) + E_4(BA_4)\right]$$

$$C_4 = \frac{1}{CA_1}\left[1 - D_4(CA_3) - E_4(CA_4)\right]$$

$$D_4 = \frac{1}{DA_1}\left[1 - DA_4 + E_3(DA_6)\right]$$

$$E_4 = \frac{1}{EA_1}\left[1 + EA_5 + \frac{EA_6(DA_6)}{DA_1}\right]$$

$$EA_5 = \frac{BA_2(S_4 - S_5)}{(BA_1)(CA_1)} + \frac{S_5^2 - S_4^2}{CA_1}$$

And finally for B5 to E5:

$$B_5 = \frac{1}{BA_1}\left[C_5(BA_2) + D_5(BA_3) + E_5(BA_4)\right],$$

$$C_5 = -\frac{1}{CA_1}\left[D_5(CA_3) + E_5(CA_4)\right],$$

$$D_5 = -\frac{E_5(DA_6)}{DA_1}, \quad E_5 = \frac{1}{EA_1}$$

A.2. Constants for symmetrical shape functions:

Let $S_1' = a_1' + 2a_2' + a_3'$ and $S_2' = a_2' + a_3'$, then:

$$B_1' = \frac{S_2'^2}{2S_1'\left(S_1'^2 - S_2'^2\right)},\ B_2' = \frac{S_1'^2}{2S_2'\left(S_1'^2 - S_2'^2\right)}$$

$$C_1' = \frac{-S_2'^2}{2S_1'^2\left(S_1'^2 - S_2'^2\right)}, C_2' = \frac{-S_1'^2}{2S_2'^2\left(S_1'^2 - S_2'^2\right)}, C_3' = \frac{-\left(S_1'^2 + S_2'^2\right)}{S_1'^2 S_2'^2}$$

$$D_1' = \frac{1}{2S_1'\left(S_1'^2 - S_2'^2\right)},\ D_2' = \frac{1}{2S_2'^2\left(S_1'^2 - S_2'^2\right)}$$

$$E_1' = \frac{1}{2S_1'^2\left(S_1'^2 - S_2'^2\right)},\ E_2' = \frac{1}{2S_2'^2\left(S_1'^2 - S_2'^2\right)},\ E_3' = \frac{1}{S_1'^4}$$

APPENDIX B

The required partial differential equations for I_5 are as follows:

$$I_{5,x} = 4x^3 B_{11} + x^4 B_{11,x} - 12x^2 B_{12} - 4x^3 B_{12,x} + 12xB_{21} + 6x^2 B_{21,x} - 4B_{22} - 4xB_{22,x} + B_{23,x}$$

$$I_{5,xy} = 4x^3 B_{11,y} + x^4 B_{11,xy} - 12x^2 B_{12,y} - 4x^3 B_{12,xy} + 12xB_{21,y} + 6x^2 B_{21,xy} - 4B_{22,y} - 4xB_{22,xy} + B_{23,xy}$$

$$I_{5,yy} = x^4 B_{11,yy} - 4x^3 B_{12,yy} + 6x^2 B_{21,yy} - 4xB_{22,yy} + B_{23,yy}$$

$$I_{5,xyy} = 4x^3 B_{11,yy} + x^4 B_{11,xyy} - 12x^2 B_{12,yy} - 4x^3 B_{12,xyy} + 12xB_{21,yy} + 6x^2 B_{21,xyy} - 4B_{22,yy} - 4xB_{22,xyy} + B_{23,xyy}$$

$$I_{5,yyy} = x^4 B_{11,yyy} - 4x^3 B_{12,yyy} + 6x^2 B_{21,yyy} - 4xB_{22,yyy} + B_{23,yyy}$$

Where constant integrals B11 and its derivatives are:

$$B_{11} = \int_{x-a}^{x+a} \ln\sqrt{z^2 + y^2}\,dz = y(\theta_1 - \theta_2) - (x-a)\ln r_1 + (x+a)\ln r_2 - 2a$$

Let $\theta_1 = Arc\tan\left(\frac{y}{x-a}\right),\ \theta_2 = Arc\tan\left(\frac{y}{x+a}\right)$

$$r_1 = \sqrt{(x-a)^2 + y^2},\ r_2 = \sqrt{(x+a)^2 + y^2}$$

$$B_{11,x} = \ln r_1 - \ln r_2, B_{11,y} = \theta_1 - \theta_2$$

$$B_{11,xy} = -\left(\frac{y}{r_1^2} - \frac{y}{r_2^2}\right)$$

$$B_{11,yy} = -B_{11,xx} = \frac{(x-a)}{r_1^2} - \frac{(x+a)}{r_2^2}$$

$$B_{11,xxx} = -B_{11,xyy} = \frac{(x-a)^2 - y^2}{r_1^4} - \frac{(x+a)^2 + y^2}{r_2^4}$$

The integral B12 and its derivatives:

$$B_{12} = \int_{x-a}^{x+a} z\ln\sqrt{z^2 + y^2}\,dz = -\frac{1}{2}\left[r_1^2\ln r_1 - r_2^2\ln r_2\right] - ax = -\frac{x^2 - y^2 + a^2}{2}(\ln r_1 - \ln r_2) + ax(\ln r_1 + \ln r_2) - ax$$

Let $B_{10} = \frac{x^2 - y^2 + a^2}{2}$ *and* $L_x = \ln r_1 - \ln r_2$ then the derivation $L_{x,x}$, $L_{x,y}$, $L_{x,xy}$, $L_{x,yy}$ etc. are:

$$L_{x,x} = -\left(\frac{(x-a)}{r_1^2} + \frac{(x+a)}{r_2^2}\right), L_{x,y} = -y\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)$$

$$L_{x,yy} = \frac{(x-a)^2 - y^2}{r_1^4} - \frac{(x+a)^2 - y^2}{r_2^4}$$

$$L_{x,xyy} = -2y\left[\frac{3(x-a)^2 - y^2}{r_1^6} + \frac{3(x+a)^2 - y^2}{r_2^6}\right]$$

$$L_{x,yyy} = -2\left[\frac{(x-a)\left(r_1^2 - 4y^2\right)}{r_1^6} + \frac{(x+a)\left(r_2^2 - 4y^2\right)}{r_2^6}\right]$$

And the derivation of $B_{12,x}$, $B_{12,y}$, $B_{12,xy}$, $B_{12,yy}$, etc, can be defined as

$$B_{12,x} = -xB_{11,x} + B_{10}B_{11,yy} + aL_x + axL_{x,x} - a$$

$$B_{12,y} = yB_{11,x} - B_{10}B_{11,xy} + axL_{x,y}$$

$$B_{12,xy} = -xB_{11,xy} - yB_{11,yy} + B_{10}B_{11,yyy} - aL_{x,y} - axL_{x,xy}$$

$$B_{12,yy} = B_{11,xx} + 2yB_{11,xy} + B_{10}B_{12,xyy} + axL_{x,yy}$$

$$B_{12,xyy} = -xB_{11,xyyy} - B_{11,yy} - 2yB_{11,yyy} + B_{10}B_{11,yyyy} + aL_{x,yy} + axL_{x,xyy}$$

$$B_{12,yyy} = 3B_{11,xy} + 3yB_{11,xyy} - B_{10}B_{11,xyyy} + axL_{x,yyy}$$

The integral B_{21} and its derivatives

$$B_{21} = \int_{x-a}^{x+a} z\ \ln\sqrt{z^2 + y^2}\,dz = -\frac{1}{3}\left[(x-a)^3\ln r_1 - (x+a)^3\ln r_2\right] - \frac{1}{3}y^3(\theta_1 - \theta_2) - \frac{2}{3}a\left(x^2 - y^2\right) - \frac{2}{9}a^3$$

This integral can be described as:

$$B_{21} = -x\left(\frac{x^2}{3}+a^2\right)B_{11,x} - \frac{1}{3}y^3 B_{11,y} + a\left(x^2+\frac{a^2}{3}\right)L_x - \frac{2}{3}a\left(x^2-y^2+\frac{a^2}{3}\right)$$

$$B_{21,x} = -x\left(x^2+a^2\right)B_{11,x} + \left(\frac{x^3}{3}+a^2x\right)B_{11,yy} - \frac{1}{3}y^3 B_{11,xy}$$

$$+2axL_x + a\left(x^2+\frac{a^2}{3}\right)L_{x,x} - \frac{4}{3}ax$$

$$B_{21,y} = -x\left(\frac{x^2}{3}+a^2\right)B_{11,xy} - \frac{1}{3}y^3 B_{11,yy}$$

$$-y^2 B_{11,y} + a\left(x^2+\frac{a^2}{3}\right)L_{x,y} + \frac{4}{3}ay$$

$$B_{21,xy} = -x\left(x^2+a^2\right)B_{11,xy} + \left(\frac{x^3}{3}+a^2x\right)B_{11,xxy} - \frac{1}{3}y^3 B_{11,xyy}$$

$$-y^2 B_{11,xy} + 2axL_{x,y} + a\left(x^2+\frac{a^2}{3}\right)L_{x,xy}$$

$$B_{21,yy} = -x\left(\frac{x^2}{3}+a^2\right)B_{11,xyy} - \frac{1}{3}y^3 B_{11,yyy} - 2y^2 B_{11,y}$$

$$-2yB_{11,y} + a\left(x^2+\frac{a^2}{3}\right)L_{x,yy} + \frac{4}{3}a$$

APPENDIX C

Constants defining crack tip shape functions:

Constants for N_{C1}:

$$C''_{11} = \frac{C_{11}}{C_{40}} - \frac{C_{14}C_{41}}{C_{44}} + \left(C_{15} - \frac{C_{14}C_{45}}{C_{44}}\right)C''_{51},$$

$$C''_{21} = C_{21} - \frac{C_{24}C_{41}}{C_{44}} + \left(C_{25} - \frac{C_{24}C_{45}}{C_{44}}\right)C''$$

$$C''_{31} = C_{31} - \frac{C_{34}C_{41}}{C_{44}} + \left(C_{35} - \frac{C_{34}C_{45}}{C_{44}}\right)C''_{51},$$

$$C''_{41} = -\frac{C_{41}}{C_{40}} - \frac{C_{45}}{C_{44}}C''_{51},\ C''_{51} = \frac{C_{501}}{1-C_{500}}$$

Constants for N_{C2}:

$$C''_{12} = \frac{C_{12}}{C_{40}} - \frac{C_{14}C_{42}}{C_{44}} + \left(C_{15} - \frac{C_{14}C_{45}}{C_{44}}\right)C''_{52},$$

$$C''_{22} = C_{22} - \frac{C_{24}C_{42}}{C_{44}} + \left(C_{25} - \frac{C_{24}C_{45}}{C_{44}}\right)C''_{52}$$

$$C''_{32} = C_{32} - \frac{C_{34}C_{42}}{C_{44}} + \left(C_{35} - \frac{C_{34}C_{45}}{C_{44}}\right)C''_{52},$$

$$C''_{42} = -\frac{C_{42}}{C_{40}} - \frac{C_{45}}{C_{44}}C''_{52}, C''_{52} = \frac{C_{502}}{1-C_{500}}$$

Constants for N_{C3}:

$$C''_{13} = \frac{C_{13}}{C_{40}} - \frac{C_{14}C_{43}}{C_{44}} + \left(C_{15} - \frac{C_{14}C_{45}}{C_{44}}\right)C''_{53},$$

$$C''_{23} = C_{23} - \frac{C_{24}C_{43}}{C_{44}} + \left(C_{25} - \frac{C_{24}C_{45}}{C_{44}}\right)C''_{53}$$

$$C''_{33} = C_{33} - \frac{C_{34}C_{43}}{C_{44}} + \left(C_{35} - \frac{C_{34}C_{45}}{C_{44}}\right)C''_{53},$$

$$C''_{43} = -\frac{C_{43}}{C_{40}} - \frac{C_{45}}{C_{44}}C''_{53},\ C''_{53} = \frac{C_{503}}{1-C_{500}}$$

Constants for N_{C4}:

$$C''_{14} = \frac{C_{14}C_{40}}{\sqrt{7}C_{44}} + \left(C_{15} - \frac{C_{14}C_{45}}{C_{44}}\right)C''_{54},$$

$$C''_{24} = \frac{C_{24}C_{40}}{\sqrt{7}C_{44}} + \left(C_{25} - \frac{C_{24}C_{45}}{C_{44}}\right)C''_{54}$$

$$C''_{34} = \frac{C_{34}C_{40}}{\sqrt{7}C_{44}} + \left(C_{35} - \frac{C_{34}C_{45}}{C_{44}}\right)C''_{54},$$

$$C''_{44} = \frac{C_{40}}{\sqrt{7}C_{44}} - \frac{C_{45}}{C_{44}}C''_{54}, C''_{54} = \frac{C_{504}}{1-C_{500}}$$

Constants for N_{C5}:

$$C''_{15} = \left(C_{15} - \frac{C_{14}C_{45}}{C_{44}}\right)C''_{55},$$

$$C''_{25} = \left(C_{25} - \frac{C_{24}C_{45}}{C_{44}}\right)C''_{55},$$

$$C''_{35} = \left(C_{35} - \frac{C_{34}C_{45}}{C_{44}}\right)C''_{55},$$

$$C''_{45} = -\frac{C_{45}}{C_{44}}C''_{55}\ C''_{55} = \frac{C_{505}}{1-C_{500}},$$

$$C''_{55} = \frac{C_{505}}{1-C_{500}}$$

where

$$C_{505} = (9a_1)^{-9/2}$$

APPENDIX D

$$u_i(\xi) = C_{1c}\xi^{1/2} + C_{2c}\xi^{3/2} + C_{3c}\xi^{5/2} + C_{4c}\xi^{7/2} + C_{5c}\xi^{9/2}$$

$$D_{ic}(\xi)=N_{1c}(\xi)D_{ic}^{1}+N_{2c}(\xi)D_{ic}^{2}+N_{3c}(\xi)D_{ic}^{3}+N_{4c}(\xi)D_{ic}^{4}+N_{5c}(\xi)D_{ic}^{5}$$

$$D_{ic}(a_1)=C_{1c}a_1^{1/2}+C_{2c}a_1^{3/2}+C_{3c}a_1^{5/2}+C_{4c}a_1^{7/2}+C_{5c}a_1^{9/2}$$

$$D_{ic}(2a_1+a_2)=C_{1c}(2a_1+a_2)^{1/2}+C_{2c}(2a_1+a_2)^{3/2}+C_{3c}(2a_1+a_2)^{5/2}+C_{4c}(2a_1+a_2)^{7/2}+C_{5c}(2a_1+a_2)^{9/2}$$

$$D_{ic}(2a_1+2a_2+a_3)=C_{1c}(2a_1+2a_2+a_3)^{1/2}+C_{2c}(2a_1+2a_2+a_3)^{3/2}+C_{3c}(2a_1+2a_2+a_3)^{5/2}+C_{4c}(2a_1+2a_2+a_3)^{7/2}+C_{5c}(2a_1+a_2)^{9/2}$$

$$D_{ic}(2a_1+2a_2+2a_3+a_4)=C_{1c}(2a_1+2a_2+2a_3+a_4)^{1/2}+C_{2c}(2a_1+2a_2+2a_3+a_4)^{3/2}+C_{3c}(2a_1+2a_2+2a_3+a_4)^{5/2}+C_{4c}(2a_1+2a_2+2a_3+a_4)^{7/2}+C_{5c}(2a_1+2a_2+2a_3+a_4)^{9/2}$$

$$D_{ic}(2a_1+2a_2+2a_3+2a_4+a_5)=C_{1c}(2a_1+2a_2+2a_3+2a_4+a_5)^{1/2}+C_{2c}(2a_1+2a_2+2a_3+2a_4+a_5)^{3/2}+C_{3c}(2a_1+2a_2+2a_3+2a_4+a_5)^{5/2}+C_{4c}(2a_1+2a_2+2a_3+2a_4+a_5)^{7/2}+C_{5c}(2a_1+2a_2+2a_3+2a_4+a_5)^{9/2}$$

Assuming equal sub-elements $a_1=a_2=a_3=a_4=a_5$**, this one may be obtained.**

$$D_{ic}^{1}(a_1)=C_{1c}a_1^{1/2}+C_{2c}a_1^{3/2}+C_{3c}a_1^{5/2}+C_{4c}a_1^{7/2}+C_{5c}a_1^{9/2}$$

$$D_{ic}^{2}(3a_1)=C_{1c}(3a_1)^{1/2}+C_{2c}(3a_1)^{3/2}+C_{3c}(3a_1)^{5/2}+C_{4c}(3a_1)^{7/2}+C_{5c}(3a_1)^{9/2}$$

$$D_{ic}^{3}(5a_1)=C_{1c}(5a_1)^{1/2}+C_{2c}(5a_1)^{3/2}+C_{3c}(5a_1)^{5/2}+C_{4c}(5a_1)^{7/2}+C_{5c}(5a_1)^{9/2}$$

$$D_{ic}^{4}(7a_1)=C_{1c}(7a_1)^{1/2}+C_{2c}(7a_1)^{3/2}+C_{3c}(7a_1)^{5/2}+C_{4c}(7a_1)^{7/2}+C_{5c}(7a_1)^{9/2}$$

$$D_{ic}^{5}(9a_1)=C_{1c}(9a_1)^{1/2}+C_{2c}(9a_1)^{3/2}+C_{3c}(9a_1)^{5/2}+C_{4c}(9a_1)^{7/2}+C_{5c}(9a_1)^{9/2}$$

Let $s_{1c}=a_1$, $s_{2c}=2a_1+a_2$, $s_{3c}=2(a_1+a_2)+a_3$, $s_{4c}=2(a_1+a_2+a_3)+a_4$ **and** $S_{5c}=2(a_1+a_2+a_3+a_4)+a_5$, **then**

$$D_{ic}^{1}(s_{1c})=C_{1c}s_{1c}^{1/2}+C_{2c}s_{1c}^{3/2}+C_{3c}s_{1c}^{5/2}+C_{4c}s_{1c}^{7/2}+C_{5c}s_{1c}^{9/2}$$

$$sc_{31}=\frac{s_{2c}-s_{3c}}{s_{1c}^{1/2}(s_{1c}-s_{2c})\left(s_{3c}^2-s_{1c}s_{2c}-s_{3c}s_{1c}-s_{3c}s_{2c}\right)}$$

$$sc_{32}=\frac{s_{3c}-s_{1c}^{1/2}s_{2c}^{1/2}}{s_{2c}^{1/2}(s_{1c}-s_{2c})sc_{30}},\quad sc_{33}=\frac{1}{s_{3c}^{1/2}sc_{30}}$$

$$sc_{34}=\frac{s_{3c}s_{1c}^2+s_{1c}s_{2c}s_{3c}+s_{3c}s_{2c}^2-s_{1c}^2s_{2c}-s_{2c}^2s_{1c}-s_{3c}^3}{sc_{30}}$$

$$sc_{35}=\frac{1}{sc_{30}}(s_{3c}s_{1c}^3+s_{1c}^2s_{2c}s_{3c}+s_{1C}s_{2c}^2s_{3c}+s_{2c}^3s_{3c}-s_{1c}^3s_{2c}-s_{1c}^2s_{2c}^2-s_{1c}s_{2c}^3-s_{3c}^4)$$

$$sc_{11}=\frac{-s_{2c}}{s_{1c}^{1/2}(s_{1c}-s_{3c})}+s_{1c}s_{2c}sc_{31},$$

$$sc_{12}=\frac{s_{1c}^{1/2}}{s_{1c}-s_{2c}}+s_{1c}s_{2c}sc,\quad sc_{13}=s_{1c}s_{2c}sc_{33},$$

$$sc_{14}=s_{1c}s_{2c}sc_{34}+s_{1c}^2s_{2c}+s_{1c}s_{2c}^2,$$

$$sc_{15}=s_{1c}s_{2c}sc_{35}+s_{1c}^3s_{2c}+s_{1c}s_{2c}^3+s_{1c}^2s_{2c}^2$$

$$sc_{21}=\frac{1}{\sqrt{s_{1c}}(s_{1c}-s_{2c})}-(s_{1c}+s_{2c})sc_{31},$$

$$sc_{22}=\frac{-1}{\sqrt{s_{2c}}(s_{1c}-s_{2c})}-(s_{1c}+s_{2c})sc_{32},$$

$$sc_{23}=-(s_{1c}+s_{2c})sc_{33}$$

$$sc_{24}=-s_{1c}^2-s_{1c}s_{2c}-s_{2c}^2-(s_{1c}+s_{2c})sc_{34},$$

$$sc_{25}=-s_{1c}^3-s_{1c}^2s_{2c}-s_{1c}s_{2c}^2-s_{2c}^3-(s_{1c}+s_{2c})sc_{35}$$

$$sc_{40}=sc_{14}+sc_{24}s_{4c}+sc_{34}s_{4c}^2+s_{4c}^3$$

$$sc_{41}=\frac{-sc_{11}-sc_{21}s_{4c}-sc_{31}s_{4c}^2}{sc_{40}}=C'_{41},$$

$$sc_{42}=\frac{-sc_{12}-sc_{22}s_{4c}-sc_{32}s_{4c}^2}{sc_{40}}=C'_{42},$$

$$sc_{43}=\frac{-sc_{13}-sc_{23}s_{4c}-sc_{33}s_{4c}^2}{sc_{40}}=C'_{43},$$

$$sc_{44}=\frac{1}{\sqrt{s_{4c}}\,sc_{40}}=C'_{44}$$

$$sc_{45}=\frac{-sc_{15}-sc_{25}s_{4c}-sc_{35}s_{4c}^2-s_{4c}^4}{sc_{40}}=C'_{45}$$

$$C'_{11}=sc_{11}+sc_{14}sc_{41},\quad C'_{12}=sc_{12}+sc_{14}sc_{42},$$

$$C'_{13}=sc_{13}+sc_{14}sc_{43},\quad C'_{14}=sc_{14}sc_{44},$$

$$C'_{15}=sc_{15}+sc_{14}sc_{45}$$

$$C'_{21}=sc_{21}+sc_{24}sc_{41},\quad C'_{22}=sc_{22}+sc_{24}sc_{42},$$

$$C'_{23}=sc_{23}+sc_{24}sc_{43},\quad C'_{24}=sc_{24}sc_{44},$$

$C'_{25} = sc_{25} + sc_{24}sc_{45}$

$C'_{31} = sc_{31} + sc_{34}sc_{41}$, $C'_{32} = sc_{32} + sc_{34}sc_{42}$,

$C'_{33} = sc_{33} + sc_{34}sc_{43}$, $C'_{34} = sc_{34}sc_{44}$,

$C'_{35} = sc_{35} + sc_{34}sc_{45}$

$C_{50} = C'_{15} + s_{5c}C'_{25} + s_{5c}^2 C'_{35} + s_{5c}^2 sc_{45} + s_{5c}^4$

$$C_{51} = \frac{-C'_{11} - C'_{21}s_{5c} - C'_{31}s_{5c}^2 - sc_{41}s_{5c}^3}{C_{50}},$$

$$C_{52} = \frac{-C'_{12} - C'_{22}s_{5c} - C'_{32}s_{5c}^2 - sc_{42}s_{5c}^3}{C_{50}}$$

$$C_{53} = \frac{-C'_{13} - C'_{23}s_{5c} - C'_{33}s_{5c}^2 - sc_{43}s_{5c}^3}{C_{50}},$$

$$C_{54} = \frac{-C'_{14} - C'_{24}s_{5c} - C'_{34}s_{5c}^2 - sc_{44}s_{5c}^3}{C_{50}}$$

$$C_{52} = \frac{1}{\sqrt{s_{5c}C_{50}}}$$

$C_{11} = C'_{11} + C'_{15}C_{51}$, $C_{12} = C'_{12} + C'_{15}C_{52}$,

$C_{13} = C'_{13} + C'_{15}C_{53}$, $C_{14} = C'_{14} + C'_{15}C_{54}$,

$C_{15} = C'_{15}C_{55}$

$C_{21} = C'_{21} + C'_{25}C_{51}$, $C_{22} = C'_{22} + C'_{25}C_{52}$

$C_{23} = C'_{23} + C'_{25}C_{53}$, $C_{24} = C'_{24} + C'_{25}C_{54}$,

$C_{25} = C'_{25}C_{55}$

$C_{31} = C'_{31} + C'_{35}C_{51}$, $C_{32} = C'_{32} + C'_{35}C_{52}$,

$C_{33} = C'_{33} + C'_{35}C_{53}$, $C_{34} = C'_{34} + C'_{35}C_{54}$,

$C_{35} = C'_{35}C_{55}$

$C_{41} = sc_{41} + sc_{45}C_{51}$, $C_{42} = sc_{42} + sc_{45}C_{52}$,

$C_{43} = sc_{43} + sc_{45}C_{53}$, $C_{44} = sc_{44} + sc_{45}C_{54}$,

$C_{45} = sc_{45}C_{55}$

Substituting following equations

$$C_{1c} = C_{11}D_{ic}^1 + C_{12}D_{ic}^2 + C_{13}D_{ic}^3 + C_{14}D_{ic}^4 + C_{15}D_{ic}^5$$

$$C_{2c} = C_{21}D_{ic}^1 + C_{22}D_{ic}^2 + C_{23}D_{ic}^3 + C_{24}D_{ic}^4 + C_{25}D_{ic}^5$$

$$C_{3c} = C_{31}D_{ic}^1 + C_{32}D_{ic}^2 + C_{33}D_{ic}^3 + C_{34}D_{ic}^4 + C_{35}D_{ic}^5$$

$$C_{4c} = C_{41}D_{ic}^1 + C_{42}D_{ic}^2 + C_{43}D_{ic}^3 + C_{44}D_{ic}^4 + C_{45}D_{ic}^5$$

$$C_{5c} = C_{51}D_{ic}^1 + C_{52}D_{ic}^2 + C_{53}D_{ic}^3 + C_{54}D_{ic}^4 + C_{55}D_{ic}^5$$

Into

$$D_{ic}^1(\xi) = C_{1c}\xi^{1/2} + C_{2c}\xi^{3/2} + C_{3c}\xi^{5/2} + C_{4c}\xi^{7/2} + C_{5c}\xi^{9/2}$$

the following shape functions will emerge

$$N_{1c}(\xi) = C_{11}\xi^{1/2} + C_{21}\xi^{3/2} + C_{31}\xi^{5/2} + C_{41}\xi^{7/2} + C_{51}\xi^{9/2}$$

$$N_{2c}(\xi) = C_{12}\xi^{1/2} + C_{22}\xi^{3/2} + C_{32}\xi^{5/2} + C_{42}\xi^{7/2} + C_{52}\xi^{9/2}$$

$$N_{3c}(\xi) = C_{13}\xi^{1/2} + C_{32}\xi^{3/2} + C_{33}\xi^{5/2} + C_{43}\xi^{7/2} + C_{53}\xi^{9/2}$$

$$N_{4c}(\xi) = C_{14}\xi^{1/2} + C_{24}\xi^{3/2} + C_{34}\xi^{5/2} + C_{44}\xi^{7/2} + C_{54}\xi^{9/2}$$

$$N_{5c}(\xi) = C_{15}\xi^{1/2} + C_{25}\xi^{3/2} + C_{35}\xi^{5/2} + C_{45}\xi^{7/2} + C_{55}\xi^{9/2}$$

And finally:

$$D_{ic}(\xi) = N_{1c}(\xi)D_{ic}^1 + N_{2c}(\xi)D_{ic}^2 + N_{3c}(\xi)D_{ic}^3 + N_{4c}(\xi)D_{ic}^4 + N_{5c}(\xi)D_{ic}^5$$