Journal of Statistical Modelling: Theory and Applications
Vol. 4, No. 2, 2023, pp. 131-158
Yazd University Press 2023

JSMTA

Research Paper

Linear inference for order statistics of Lindley distribution

Adeleh Fallah*
Department of Statistics, University of Payame Noor, Tehran, Iran.

Received: April 27, 2024/ Revised: August 09, 2024/ Accepted: September 09, 2024

Abstract: In this paper, order statistics and associated inferences are considered from Lindley distribution. We derive the exact forms of means, variances and covariances as well as the moment generating functions of order statistics. These obtained forms allow us to compute the means, variances, and covariances of the order statistics for various values of the shape parameter. These values are then used to compute the coefficients of the best linear unbiased estimators, the best linear invariant estimators, and the least square estimators of the location and scale parameters. The variances and covariances of these estimators are also presented. Using the best linear unbiased estimators and best linear invariant estimators we construct confidence intervals for the location and scale parameters through Monte Carlo simulations. In addition, based on the ordered data, we investigate how to obtain the best linear unbiased predictor and the best linear invariant predictor for future order statistics. Finally, data analysis and Monte Carlo simulation have been performed for illustrative purposes and comparative studies, respectively.
Keywords: Best linear invariant estimators; Best linear unbiased estimator; Least square estimator; Lindley distribution; Moments of order statistics; Prediction.
Mathematics Subject Classification (2010): 62F15, 62F25, 68U20.

1 Introduction

Consider the Lindley distribution with the probability density function (pdf)

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x > 0, \quad \theta > 0, \tag{1}$$

*Corresponding author: adelehfallah@pnu.ac.ir.

and the cumulative distribution function (cdf)

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}, \qquad x > 0, \quad \theta > 0, \tag{2}$$

where θ is the shape parameter. The Lindley (LI) distribution was introduced by Lindley (1958) in the context of Fiducial and Bayesian statistics. The Lindley pdf can be written as a mixture of an exponential distribution with scale parameter θ and a gamma $G(2, \lambda)$ with mixing proportion $p = \theta/(\theta + 1)$. The LI distribution received considerable attention in the last few years. Ghitany et al. (2008) showed that in many ways the LI distribution can fit the data effectively better than the exponential model. They studied various distributional properties and other related issues of this distribution. For further studies about LI distribution, one may refer to Chesneau et al. (2021), Kharazmi et al. (2023), Ali et al. (2013), Nadarajah et al. (2011), Krishna and Kumar (2011), Bakouch et al. (2012), Zakerzadeh and Dolati (2009), Gupta and Singhn (2013), Al-Mutairi et al. (2013), Gomez-Deniz et al. (2014) and Asgharzadeh et al. (2016).

Order statistics have found important applications including life testing, reliability theory, characterization, statistical quality control, signal and image processing and many other fields. Order statistics and related inferences have been discussed for many distributions, see for example Arnold et al. (2008) and Balakrishnan and Cohen (1990). However, to the best of our knowledge, LI order statistics and their use in the inference are not considered so far in the literature. For this, our main aim in this paper is to consider and study the problem of ordered data of the LI distribution. In this context, we derive exact expressions of all moments of LI order statistics. The means, variances and covariances of LI order statistics are readily computed based on the resulting expressions. Then, we use these moments to obtain best linear unbiased estimators (BLUEs) for the location and scale parameters. The prediction of future order statistics based on Type II censoring sample from 3-parameter LI distribution is also discussed.

Regarding the best linear unbiased estimations we mention, for example, Kumar et al. (2023) who have discussed Estimation of the location and scale parameters of generalized Pareto distribution based on progressively type-II censored order statistics. Mahmoud et al. (2003) have obtained the best linear unbiased estimates (BLUEs) and interval of future order statistics from inverse Weibull distribution by using Type-II censoring. MirMostafaee et al. (2016) have derived the single and product moments of Record values from NH distribution. Then, they have used these moments to obtain the BLUEs for the location and scale parameters.

The rest of the paper is organized as follows. In Section 2, we derive the means, variances, covariances and moment generating functions of order statistics. Next, in Section 3, we obtain the BLUEs, best linear invariant estimators (BLIEs) and least square estimators (LSEs) of the location and scale parameters. The BLUEs and BLIEs are then used to construct confidence intervals (CIs) for the location and scale parameters. In Section 4, we discuss point predictions for future order statistics. Finally, we present two numerical examples to illustrate the results developed in this paper and conduct a numerical simulation study to assess the efficiencies of the BLUEs with respect to the MLEs, BLIEs and LSEs of the location and scale parameters.

2 Moments of order statistics

In this section, we present exact expressions of the single and product moments as well as moment generating function of LI order statistics. The expressions presented here are explicit and can be used effectively in practical set-ups.

2.1 Single moments

Let $X_1, X_2, \ldots, X_n$ be independent, absolutely continuous random variables with common pdf $f(x)$ and cdf $F(x)$, and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. The marginal density function of $X_{i:n}$ $(i = 1, 2, \ldots, n)$ is given by (Balakrishnan and Cohen, 1990)

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x), \qquad -\infty < x < \infty.$$

Further, the kth moment of $X_{i:n}$ $(i = 1, 2, \ldots, n)$ can be written as (see Balakrishnan and Cohen (1990), P. 25)

$$\begin{aligned}
\alpha_{i:n}^{(k)} &= E(X_{i:n}^k) \\
&= \frac{n!}{(i-1)!(n-i)!} \int_0^\infty x^k [F(x)]^{i-1}[1-F(x)]^{n-i} f(x)dx \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \int_0^\infty x^k [1-F(x)]^{n+j-i} f(x)dx. \qquad (3)
\end{aligned}$$

By plugging the expressions of the pdf and cdf in (1) and (2) respectively, into (3), we obtain

$$\begin{aligned}
\alpha_{i:n}^{(k)} &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \int_0^\infty x^k \left(\frac{\theta+1+\theta x}{\theta+1} \, e^{-\theta x} \right)^{n+j-i} \\
&\quad \times \left(\frac{\theta^2}{\theta+1} \right) (1+x) e^{-\theta x} dx \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j g_1(j, n, i, k, \theta),
\end{aligned}$$

where

$$\begin{aligned}
g_1(j, n, i, k, \theta) &= \frac{\theta^2}{(\theta+1)} \int_0^\infty \left(\frac{\theta+1+\theta x}{\theta+1} \, e^{-\theta x} \right)^{n+j-i} x^k (1+x) \, e^{-\theta x} dx \\
&= \frac{\theta^2}{(\theta+1)} \int_0^\infty \left(1 + \frac{\theta x}{\theta+1} \right)^{n+j-i} e^{-\theta(n+j-i+1)x} \, x^k (1+x) dx.
\end{aligned}$$

By making the transformation $y = \frac{\theta x}{\theta+1}$, we have then

$$g_1(j, n, i, k, \theta) = \frac{(\theta+1)^k}{\theta^{k-1}} \int_0^\infty y^k \left(1 + \frac{\theta+1}{\theta} y \right) (1+y)^{n+j-i} \, e^{-[(n+j-i+1)(\theta+1)]y} dy$$

$$
\begin{aligned}
&= \frac{(\theta+1)^k}{\theta^{k-1}} \sum_{r=0}^{n+j-i} \binom{n+j-i}{r} \int_0^\infty y^{n+j-i-r+k} \left(1 + \frac{(\theta+1)}{\theta} y\right) \\
&\times \exp\{-(n+j-i+1)(\theta+1)y\} dy \\
&= \frac{(\theta+1)^{k+1}}{\theta^k} \sum_{r=0}^{n+j-i} \binom{n+j-i}{r} \\
&\times \left(\frac{\theta(n+j-i+1)\ (n+j-i-r+k)! + (n+j-i-r+k+1)!}{[(n+j-i+1)(\theta+1)]^{n+j-i-r+k+2}} \right).
\end{aligned}
$$

Now, from (3), the first and second moments of order statistics $X_{i:n}$ $(i = 1, 2, \ldots, n)$, are respectively;

$$
\begin{aligned}
\alpha_{i:n} = E(X_{i:n}) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \int_0^\infty x \left(\frac{\theta+1+\theta x}{\theta+1}\ e^{-\theta x}\right)^{n+j-i} \\
&\times \left(\frac{\theta^2}{\theta+1}\right) (1+x) e^{-\theta x} dx \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \sum_{r=0}^{n+j-i} \binom{i-1}{j} \binom{n+j-i}{r} (-1)^j \frac{(\theta+1)^2}{\theta} \\
&\times \left(\frac{\theta(n+j-i+1)\ (n+j-i-r+1)! + (n+j-i-r+2)!}{[(n+j-i+1)(\theta+1)]^{n+j-i-r+3}} \right), \\
\alpha^{(2)}_{i:n} = E(X^2_{i:n}) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \int_0^\infty x^2 \left(\frac{\theta+1+\theta x}{\theta+1}\ e^{-\theta x}\right)^{n+j-i} \\
&\times \left(\frac{\theta^2}{\theta+1}\right) (1+x) e^{-\theta x} dx \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \sum_{r=0}^{n+j-i} \binom{i-1}{j} \binom{n+j-i}{r} (-1)^j \frac{(\theta+1)^3}{\theta^2} \\
&\times \left(\frac{\theta(n+j-i+1)\ (n+j-i-r+2)! + (n+j-i-r+3)!}{[(n+j-i+1)(\theta+1)]^{n+j-i-r+4}} \right).
\end{aligned}
$$

From the first two single moments of $X_{i:n}$, we may compute the variance $X_{i:n}$ as

$$
\beta_{i,i:n} = Var(X_{i:n}) = \alpha^{(2)}_{i:n} - \alpha^2_{i:n}, \qquad (1 \le i \le n).
$$

We have computed the means $\alpha_{i:n}$ $(i = 1, 2, \ldots, n)$ for $n = 2(1)8$ and $\theta = 0.5(0.5)4.5$ and these values are reported in Table 1. The variances and covariances $\beta_{i,j:n}$ $(i \le j \le n, i = 1, 2, \ldots, n)$ are reported for $n = 2(1)8$ in Tables 2-3. It worth mentioning that for the special case where $n = i = 1$, the means and variance of the LI distribution, respectively, are (Ghitany et al., 2008)

$$
\alpha_{1:1} = E(X_{1:1}) = \frac{\theta+2}{\theta(\theta+1)}, \quad \alpha^{(2)}_{1:1} = E(X^{(2)}_{1:1}) = \frac{2(\theta+3)}{\theta^2(\theta+1)}, \quad Var(X_{1:1}) = \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2}.
$$

2.2 Moment generating function

Arguments similar to those in Subsection 2.1, one can also provide exact expression for the moment generating function (MGF) of any ith order statistic (i=1, 2, ..., n). By getting an explicit expression of the MGF of $X_{i:n}$, all moments of ith order statistics follow immediately. The MGF of $X_{i:n}$ $(i = 1, 2, \ldots, n)$ is derived to be

$$\begin{aligned}
M_{i:n}(t) &= E[e^{tX_{i:n}}] \\
&= \frac{n!}{(i-1)!(n-i)!}\int_0^\infty e^{tx}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)dx \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\int_0^\infty e^{tx}[1-F(x)]^{n+j-i}f(x)dx \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\int_0^\infty e^{tx}\left(\frac{\theta+1+\theta x}{\theta+1}\ e^{-\theta x}\right)^{n+j-i} \\
&\quad \times\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}dx \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\ g_2(n,i,j,\theta),
\end{aligned}$$

where

$$\begin{aligned}
g_2(n,i,j,\theta) &= \frac{\theta^2}{(\theta+1)}\int_0^\infty\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^{n+j-i}(1+x)\ e^{-(\theta-t)x}dx \\
&= \frac{\theta^2}{(\theta+1)}\int_0^\infty\left(1+\frac{\theta x}{\theta+1}\right)^{n+j-i}e^{-[\theta(n+j-i+1)-t]x}\ (1+x)dx.
\end{aligned}$$

By using the transformation $y = \frac{\theta x}{\theta+1}$, we can write $g_2(n,i,j,\theta)$ as follow

$$\begin{aligned}
g_2(n,i,j,\theta) &= \theta\int_0^\infty(1+y)^{n+j-i}\left(1+\frac{\theta+1}{\theta}y\right) \\
&\quad \times\exp\left\{-\left[(n+j-i+1)(\theta+1)-\left(\frac{\theta+1}{\theta}\right)t\right]y\right\}dy \\
&= \theta\sum_{r=0}^{n+j-i}\binom{n+j-i}{r}\int_0^\infty y^{n+j-i-r}\left(1+\frac{\theta+1}{\theta}y\right) \\
&\quad \times\exp\left\{-\left[(n+j-i+1)(\theta+1)-\left(\frac{\theta+1}{\theta}\right)t\right]y\right\}dy \\
&= \sum_{r=0}^{n+j-i}\binom{n+j-i}{r}\frac{\theta^{n+j-i-r+2}}{(1+\theta)^{n+j-i-r+1}} \\
&\quad \times\left(\frac{[\theta(n+j-i+1)-t]\ (n+j-i-r)!+(n+j-i-r+1)!}{[\theta(n+j-i+1)-t]^{n+j-i-r+2}}\right).
\end{aligned}$$

Differentiating $M_{i:n}(t)$ with respect to t, we obtain;

$$M'_{i:n}(t) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \sum_{r=0}^{n+j-i} \binom{i-1}{j} \binom{n+j-i}{r} (-1)^j \frac{\theta^{n+j-i-r+2}}{(1+\theta)^{n+j-i-r+1}} \times \left(\frac{[\theta(n+j-i+1)-t]\ (n+j-i-r+1)! + (n+j-i-r+2)!}{[\theta(n+j-i+1)-t]^{n+j-i-r+3}} \right).$$

In fact, the explicit expressions of the 1st moments of $X_{i:n}$ $(i = 1, 2, \ldots, n)$ can be obtained directly by setting setting $t = 0$ in $M'_{i:n}(t)$, which gives the results obtained previously.

2.3 Product moments

The joint density function of $X_{i:n}$ and $X_{j:n}$ $(1 \leq i < j \leq n)$ (Balakrishnan and Cohen, 1990) is

$$f_{i,j:n}(x,y) = \frac{n!}{(i-1)!\ (j-i-1)!\ (n-j)!} [F(x)]^{i-1} [F(y)-F(x)]^{j-i-1} \times [1-F(y)]^{n-j} f(x)\ f(y), \qquad -\infty \leq x < y \leq \infty.$$

The product moments of order statistics can be written as

$$\begin{aligned} \alpha_{i,j:n} = E(X_{i:n}X_{j:n}) &= \frac{n!}{(i-1)!\ (j-i-1)!\ (n-j)!} \int_{-\infty}^{\infty} \int_{x}^{\infty} xy[F(x)]^{i-1} \\ &\quad \times [F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j} f(x)f(y)dydx \\ &= \frac{n!}{(i-1)!\ (j-i-1)!\ (n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \\ &\quad \times \binom{j-i-1}{s} \int_{-\infty}^{\infty} \int_{x}^{\infty} xy[1-F(x)]^{j-i-s+r-1} \\ &\quad \times [1-F(y)]^{n-j+s} f(x)f(y)dydx. \end{aligned}$$

For the LI distribution, we have for $(1 \leq i < j \leq n)$

$$\begin{aligned} \alpha_{i,j:n} = E(X_{i:n}X_{j:n}) &= \frac{n!}{(i-1)!\ (j-i-1)!\ (n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \\ &\quad \times \binom{j-i-1}{s} \frac{\theta^4}{(\theta+1)^2} \int_0^{\infty} x(1+x)\ e^{-\theta x} \left(\frac{\theta+1+\theta x}{\theta+1}\ e^{-\theta x} \right)^{j-i-s+r-1} \\ &\quad \times \left[\int_x^{\infty} y(1+y)\ e^{-\theta y} \left(\frac{\theta+1+\theta y}{\theta+1}\ e^{-\theta y} \right)^{n-j+s} dy \right] dx. \end{aligned} \tag{4}$$

Note that

$$\int_x^{\infty} y(1+y)\ e^{-\theta y} \left(\frac{\theta+1+\theta y}{\theta+1}\ e^{-\theta y} \right)^{n-j+s} dy$$

$$= \sum_{q_1=0}^{n-j+s} \binom{n-j+s}{q_1} \left(\frac{\theta}{\theta+1}\right)^{n-j+s-q_1} \{I_1(x)+I_2(x)\}, \quad (5)$$

where $I_i(x) = \int_x^\infty y^{n-j+s-q_1+i} e^{-\theta(n-j+s+1)y} dy$, $i=1,2$. By making the transformation $u = \theta(n-j+s+1)y$, we get

$$I_i(x) = \frac{1}{\{\theta(n-j+s+1)\}^{n-j+s-q_1+i+1}} \int_{\theta(n-j+s+1)x}^{\infty} u^{(n-j+s-q_1+i+1)-1} e^{-u} du.$$

By using the fact that the cdf $F(x)$ of the gamma distribution can be written as a partial sum of Poisson probabilities

$$F(x) = \sum_{l=\rho}^{\infty} \frac{e^{-x} x^l}{l!}.$$

It follows that the integrals $I_i(x)$ becomes

$$\begin{aligned} I_i(x) &= \frac{\Gamma(n-j+s-q_1+i+1)}{\{\theta(n-j+s+1)\}^{n-j+s-q_1+i+1}} \\ &\times \sum_{l_1=0}^{n-j+s-q_1+1} e^{-\theta(n-j+s+1)x} \frac{\{\theta(n-j+s+1)\}^{l_1} x^{l_1}}{l_1!} \\ &= \Gamma(n-j+s-q_1+i+1) \sum_{l_1=0}^{n-j+s-q_1+1} \frac{e^{-\theta(n-j+s+1)x}\ x^{l_1}}{\{\theta(n-j+s+1)\}^{n-j+s-q_1-l_1+i+1}\ l_1!}. \end{aligned}$$

By plugging (5) into (4), we get

$$\begin{aligned} \alpha_{i,j:n} = &\frac{n!}{(i-1)!\ (j-i-1)!\ (n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} \sum_{q_1=0}^{n-j+s} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s} \\ &\times \binom{n-j+s}{q_1} \frac{\theta^{n-j+s-q_1+4}}{(\theta+1)^{n-j+s-q_1+2}} \\ &\times \int_0^\infty x(1+x)\ e^{-\theta x} \left(\frac{\theta+1+\theta x}{\theta+1}\ e^{-\theta x}\right)^{j-i-s+r-1} \{I_1(x)+I_2(x)\}\, dx \\ = &\frac{n!}{(i-1)!\ (j-i-1)!\ (n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} \sum_{q_1=0}^{n-j+s} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s} \\ &\times \binom{n-j+s}{q_1} \frac{\theta^{n-j+s-q_1+4}}{(\theta+1)^{n-j+s-q_1+2}}\ K(i,j,r,s,\theta), \end{aligned}$$

where

$$\begin{aligned} K(i,j,r,s,\theta) = &\int_0^\infty x(1+x)\ e^{-\theta x} \left(\frac{\theta+1+\theta x}{\theta+1}\ e^{-\theta x}\right)^{j-i-s+r-1} \{I_1(x)+I_2(x)\} dx \\ = &\sum_{q_2=0}^{j-i-s+r-1} \binom{j-i-s+r-1}{q_2} \left(\frac{\theta}{\theta+1}\right)^{j-i-s+r-q_2-1} \end{aligned}$$

$$\times\left\{\sum_{l_1=0}^{n-j+s-q_1+1}\frac{\Gamma(n-j+s-q_1+2)}{\{\theta(n-j+s+1)\}^{n-j+s-q_1-l_1+2}\ l_1!}\right.$$
$$\times\left[\frac{\Gamma(j-i-s+r+l_1-q_2+1)}{\{\theta(n-i+r+1)\}^{j-i-s+r+l_1-q_2+1}}\right.$$
$$\left.\left.+\frac{\Gamma(j-i-s+r+l_1-q_2+2)}{\{\theta(n-i+r+1)\}^{j-i-s+r+l_1-q_2+2}}\right]\right\}$$
$$+\left\{\sum_{l_2=0}^{n-j+s-q_1+2}\frac{\Gamma(n-j+s-q_1+3)}{\{\theta(n-j+s+1)\}^{n-j+s-q_1-l_2+3}\ l_2!}\right.$$
$$\times\left(\frac{\Gamma(j-i-s+r+l_2-q_2+1)}{\{\theta(n-i+r+1)\}^{j-i-s+r+l_2-q_2+1}}\right.$$
$$\left.\left.+\frac{\Gamma(j-i-s+r+l_2-q_2+2)}{\{\theta(n-i+r+1)\}^{j-i-s+r+l_2-q_2+2}}\right)\right\}.$$

From the first two single moments and the product moments, we may compute the covariance of $X_{i:n}$ and $X_{j:n}$ as

$$\beta_{i,j:n}=Cov(X_{i:n},X_{j:n})=\alpha_{i,j:n}-\alpha_{i:n}\alpha_{j:n},\qquad (1\le i<j\le n).$$

The values of covariances $\beta_{i,j:n}$ $(1\le i<j\le n)$ are computed and displayed for different values of n and θ in Tables 2-3.

3 Linear estimators

In this section, we use the means, variances and covariances computed in Section 2 to obtain the coefficients of the BLUEs, BLIEs and LSEs of μ and σ. Let $Y_{1:n},Y_{2:n},\ldots,Y_{n:n}$ be order statistics from a sample of size n from the location-scale family of the LI distribution (denoted by $LI(\mu,\sigma,\theta)$) with pdf

$$f_Y(y)=\frac{\theta^2}{\sigma^2(\theta+1)}(\sigma+y-\mu)\ \ e^{-\theta(\frac{y-\mu}{\sigma})},\quad y>\mu,\quad -\infty<\mu<\infty,\quad \sigma>0,\qquad (6)$$

where μ and σ are the location and scale parameters, respectively. Let us consider

$$X_{i:n}=\frac{Y_{i:n}-\mu}{\sigma},\qquad i=1,2,\ldots,n.$$

Then $X_{1:n},X_{2:n},\ldots,X_{n:n}$ are the corresponding order statistics from the standard LI distribution with pdf given in (1). Suppose that we observe the first n order statistics values $Y_{1:n}=y_{1:n},Y_{2:n}=y_{2:n},\ldots,Y_{n:n}=y_{n:n}$ from a population with cdf $F(y,.)$ and pdf $f(y,.)$. Given $y=(y_{1:n},y_{2:n},\ldots,y_{n:n})$, the likelihood function is given (Arnold et al., 1998) by

$$f_{1,2,\ldots n:n}(y_{1:n},y_{2:n},\ldots,y_{n:n})=n!\prod_{i=1}^{n}f(y_{i:n}),\qquad y_{1:n}\le y_{2:n}\le\cdots\le y_{n:n}.$$

Table 1: Means of order statistics for $\theta = 0.5(0.5)4.5$ and $n = 2(1)8$.

n	i	$\theta=0.5$	$\theta=1$	$\theta=1.5$	$\theta=2$	$\theta=2.5$	$\theta=3$	$\theta=3.5$	$\theta=4$	$\theta=4.5$
2	1	1.8889	0.8125	0.4933	0.3472	0.2653	0.2135	0.1781	0.1525	0.1331
2	2	4.7778	2.1875	1.3733	0.9861	0.7632	0.6197	0.5202	0.4475	0.3921
3	1	1.3525	0.5648	0.3379	0.2359	0.1793	0.1439	0.1197	0.1023	0.0892
3	2	2.9616	1.3079	0.8040	0.5697	0.4371	0.3528	0.2948	0.2527	0.2209
3	3	5.6859	2.6273	1.6580	1.1943	0.9263	0.7532	0.6330	0.5448	0.4777
4	1	1.0648	0.4350	0.2577	0.1789	0.1356	0.1085	0.0902	0.0770	0.0671
4	2	2.2157	0.9540	0.5787	0.4068	0.3106	0.2498	0.2083	0.1782	0.1556
4	3	3.7075	1.6617	1.0294	0.7327	0.5637	0.4557	0.3814	0.3272	0.2862
4	4	6.3453	2.9492	1.8675	1.3481	1.0472	0.8524	0.7168	0.6174	0.5415
5	1	0.8829	0.3546	0.2085	0.1442	0.1090	0.0872	0.0724	0.0618	0.0538
5	2	1.7922	0.7567	0.4545	0.3178	0.2418	0.1940	0.1615	0.1381	0.1204
5	3	2.8510	1.2501	0.7649	0.5404	0.4138	0.3334	0.2784	0.2384	0.2082
5	4	4.2785	1.9360	1.2057	0.8608	0.6636	0.5373	0.4500	0.3864	0.3381
5	5	6.8621	3.2025	2.0330	1.4699	1.1431	0.9312	0.7835	0.6751	0.5923
6	1	0.7566	0.2997	0.1752	0.1208	0.0912	0.0728	0.0604	0.0515	0.0449
6	2	1.5144	0.6292	0.3751	0.2612	0.1982	0.1588	0.1321	0.1128	0.0983
6	3	2.3478	1.0116	0.6133	0.4310	0.3289	0.2644	0.2204	0.1886	0.1646
6	4	3.3542	1.4886	0.9166	0.6499	0.4987	0.4024	0.3363	0.2883	0.2519
6	5	4.7406	2.1597	1.3502	0.9663	0.7461	0.6047	0.5069	0.4355	0.3812
6	6	7.2863	3.4110	2.1695	1.5707	1.2224	0.9965	0.8389	0.7230	0.6346
7	1	0.6634	0.2597	0.1511	0.1040	0.0784	0.0626	0.0519	0.0442	0.0385
7	2	1.3163	0.5396	0.3197	0.2219	0.1681	0.1345	0.1118	0.0954	0.0832
7	3	2.0097	0.8534	0.5136	0.3594	0.2736	0.2196	0.1828	0.1563	0.1363
7	4	2.7986	1.2226	0.7463	0.5264	0.4026	0.3242	0.2705	0.2316	0.2022
7	5	3.7708	1.6880	1.0443	0.7425	0.5707	0.4611	0.3857	0.3308	0.2892
7	6	5.1285	2.3484	1.4726	1.0559	0.8136	0.6621	0.5553	0.4773	0.4180
7	7	7.6460	3.5881	2.2857	1.6564	1.2901	1.0522	0.8861	0.7640	0.6707
8	1	0.5915	0.2292	0.1329	0.0913	0.0687	0.0548	0.0454	0.0387	0.0337
8	2	1.1669	0.4729	0.1930	0.1930	0.1460	0.1167	0.0969	0.0827	0.0720
8	3	1.7642	0.7398	0.3086	0.3086	0.2345	0.1880	0.1564	0.1336	0.1165
8	4	2.4189	1.0427	0.4440	0.4440	0.3387	0.2723	0.2269	0.1941	0.1694
8	5	3.1783	1.4025	0.6088	0.6088	0.4665	0.3761	0.3141	0.2691	0.2350
8	6	4.1264	1.8594	0.8226	0.8226	0.6333	0.5122	0.4286	0.3678	0.3217
8	7	5.4626	2.5114	1.1337	1.1337	0.8773	0.7121	0.5976	0.5138	0.4501
8	8	7.9579	3.7420	1.7311	1.7311	1.3491	1.1008	0.9273	0.7997	0.7022

For the location-scale family of the LI distribution (6), the likelihood function based on n order statistics takes the form

$$L = \frac{n!\ \theta^{2n}}{(\theta+1)^n \sigma^{2n}} \prod_{i=1}^{n} (\sigma + Y_{i:n} - \mu)\ \exp(-\frac{\theta}{\sigma}\sum_{i=1}^{n}(Y_{i:n}-\mu)), \qquad y > \mu. \tag{7}$$

From (7), it is easily seen that the MLE of $\tilde{\mu}_{ML} = y_{1:1}$. The MLE of $\tilde{\sigma}_{ML}$ can be obtained by maximizing (7) with respect to μ and σ.

The BLUEs of μ and σ (Arnold et al., 2008), pp. 171-179 take the forms

$$\hat{\mu}_{BLU} = \frac{1}{\Delta}\Big(\boldsymbol{\alpha^T\beta^{-1}\alpha 1^T\beta^{-1}} - \boldsymbol{\alpha^T\beta^{-1}1\alpha^T\beta^{-1}}\Big)\mathbf{Y} = \sum_{i=1}^{n} a_i Y_{i:n} \tag{8}$$

$$\hat{\sigma}_{BLU} = \frac{1}{\Delta}\Big(\boldsymbol{1^T\beta^{-1}1\alpha^T\beta^{-1}} - \boldsymbol{1^T\beta^{-1}\alpha 1^T\beta^{-1}}\Big)\mathbf{Y} = \sum_{i=1}^{n} b_i Y_{i:n}, \tag{9}$$

Table 2: Variances and covariances of order statistics for $\theta = 0.5(0.5)4.5$ and $n = 2(1)8$.

n	j	i	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1	1	2.4321	0.5273	0.2099	0.1086	0.0651	0.0429	0.0302	0.0223	0.0171
	2	1	2.0864	0.4726	0.1936	0.1020	0.0619	0.0412	0.0292	0.4475	0.0167
	2	2	8.5062	2.0273	0.8606	0.4650	0.2876	0.1939	0.1389	0.4475	0.0807
3	1	1	1.2689	0.2612	0.1007	0.0511	0.0302	0.0197	0.0138	0.0101	0.0077
	2	1	1.1036	0.2385	0.0946	0.0488	0.0292	0.0192	0.0135	0.0099	0.0076
	2	2	3.0324	0.6914	0.2835	0.1493	0.0905	0.0602	0.0426	0.0316	0.0243
	3	1	0.9799	0.2178	0.0882	0.0461	0.0279	0.0185	0.0131	0.0097	0.0074
	3	2	2.7145	0.6348	0.2652	0.1416	0.0868	0.0581	0.0414	0.0308	0.0238
	3	3	8.7692	2.1150	0.9061	0.4929	0.3064	0.2074	0.1490	0.1119	0.0869
4	1	1	0.8013	0.1581	0.0595	0.0297	0.0174	0.0113	0.0079	0.0058	0.0044
	2	1	0.7023	0.1459	0.0564	0.0286	0.0169	0.0111	0.0077	0.0057	0.0043
	2	2	1.6783	0.3684	0.1471	0.0760	0.0455	0.0300	0.0211	0.0156	0.0119
	3	1	0.6327	0.1355	0.0535	0.0275	0.0164	0.0108	0.0076	0.0056	0.0043
	3	2	1.5187	0.3431	0.1396	0.0731	0.0442	0.0293	0.0207	0.0153	0.0118
	3	3	3.2737	0.7641	0.3183	0.1694	0.1035	0.0691	0.0492	0.0366	0.0282
	4	1	0.5735	0.1253	0.0503	0.0262	0.0158	0.0104	0.0074	0.0054	0.0042
	4	2	1.3812	0.3180	0.1314	0.0696	0.0424	0.0283	0.0201	0.0149	0.0115
	4	3	2.9953	0.7112	0.3005	0.1617	0.0996	0.0670	0.0478	0.0358	0.0277
	4	4	8.8614	2.1508	0.9264	0.5061	0.3155	0.2141	0.1541	0.1159	0.0901
5	1	1	0.5608	0.1068	0.0394	0.0195	0.0113	0.0073	0.0051	0.0037	0.0028
	2	1	0.4943	0.0993	0.0376	0.0189	0.0111	0.0072	0.0050	0.0037	0.0028
	2	2	1.1022	0.2341	0.0914	0.0466	0.0277	0.0181	0.0127	0.0093	0.0071
	3	1	0.4483	0.0930	0.0360	0.0183	0.0108	0.0070	0.0049	0.0036	0.0027
	3	2	1.0024	0.2196	0.0857	0.0451	0.0270	0.0177	0.0125	0.0092	0.0070
	3	3	1.8698	0.4239	0.1727	0.0904	0.0546	0.0362	0.0255	0.0189	0.0145
	4	1	0.4121	0.0873	0.0343	0.0176	0.0105	0.0069	0.0048	0.0035	0.0027
	4	2	0.9233	0.2065	0.0835	0.0436	0.0262	0.0173	0.0122	0.0091	0.0069
	4	3	1.7278	0.3995	0.1651	0.0873	0.0531	0.0354	0.0251	0.0186	0.0143
	4	4	3.3945	0.8027	0.3377	0.1810	0.1111	0.0745	0.0531	0.0396	0.0306
	5	1	0.3786	0.0815	0.0324	0.0168	0.0101	0.0067	0.0047	0.0035	0.0027
	5	2	0.8496	0.1929	0.0790	0.0416	0.0253	0.0168	0.0119	0.0088	0.0068
	5	3	1.5944	0.3738	0.1564	0.0836	0.0512	0.0343	0.0244	0.0182	0.0141
	5	4	3.1470	0.8027	0.3208	0.1735	0.1073	0.0723	0.0518	0.0388	0.0300
	5	5	8.8932	2.1670	0.9367	0.5132	0.3207	0.2180	0.1571	0.1183	0.0920
6	1	1	0.4185	0.0773	0.0281	0.0138	0.0080	0.0051	0.0035	0.0026	0.0019
	2	1	0.3708	0.0723	0.0270	0.0134	0.0078	0.0050	0.0035	0.0025	0.0019
	2	2	0.7938	0.1637	0.0628	0.0317	0.0187	0.0121	0.0085	0.0062	0.0047
	3	1	0.3377	0.0681	0.0259	0.0130	0.0076	0.0050	0.0034	0.0025	0.0019
	3	2	0.7243	0.1544	0.0604	0.0308	0.0183	0.0119	0.0084	0.0061	0.0047
	3	3	1.2560	0.2774	0.1109	0.0573	0.0343	0.0226	0.0159	0.0117	0.0090
	4	1	0.3123	0.0644	0.0249	0.0126	0.0075	0.0049	0.0034	0.0025	0.0019
	4	2	0.6708	0.1462	0.0581	0.0299	0.0179	0.0117	0.0082	0.0061	0.0046
	4	3	1.1658	0.2630	0.1067	0.0557	0.0335	0.0222	0.0156	0.0116	0.0089
	4	4	1.9773	0.4568	0.1885	0.0996	0.0605	0.0402	0.0285	0.0211	0.0163
	5	1	0.2908	0.0610	0.0239	0.0122	0.0073	0.0048	0.0033	0.0024	0.0019
	5	2	0.6254	0.1385	0.0557	0.0289	0.0174	0.0115	0.0081	0.0060	0.0046
	5	3	1.0887	0.2493	0.1024	0.0539	0.0327	0.0217	0.0154	0.0114	0.0088
	5	4	1.8512	0.4338	0.1811	0.0965	0.0590	0.0394	0.0280	0.0208	0.0161
	5	5	3.4624	0.8255	0.3496	0.1883	0.1161	0.0780	0.0557	0.0416	0.0322
	6	1	0.2697	0.0573	0.0227	0.0117	0.0070	0.0046	0.0032	0.0024	0.0018
	6	2	0.5805	0.1301	0.0529	0.0277	0.0168	0.0111	0.0079	0.0058	0.0045
	6	3	1.0119	0.2345	0.0973	0.0517	0.0316	0.0211	0.0150	0.0112	0.0086
	6	4	1.7244	0.4087	0.1724	0.0926	0.0570	0.0383	0.0273	0.0204	0.0158
	6	5	3.2383	0.7802	0.3336	0.1811	0.1123	0.0759	0.0544	0.0408	0.0316
	6	6	8.8992	2.1743	0.9422	0.5173	0.3238	0.2204	0.1590	0.1198	0.0933
7	1	1	0.3264	0.0587	0.0211	0.0102	0.0059	0.0038	0.0026	0.0019	0.0014
	2	1	0.2905	0.0551	0.0203	0.0100	0.0058	0.0037	0.0026	0.0019	0.0014
	2	2	0.6058	0.1217	0.0460	0.0230	0.0135	0.0087	0.0061	0.0044	0.0034

Table 3: Continued Table 2.

n	j	i	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
7	3	1	0.2654	0.0522	0.0196	0.0097	0.0057	0.0037	0.0025	0.0018	0.0014
	3	2	0.5543	0.1152	0.0444	0.0224	0.0132	0.0086	0.0060	0.0044	0.0033
	3	3	0.9202	0.1984	0.0780	0.0399	0.0237	0.0155	0.0109	0.0080	0.0061
	4	1	0.2463	0.0496	0.0189	0.0095	0.0056	0.0036	0.0025	0.0018	0.0014
	4	2	0.5151	0.1096	0.0429	0.0219	0.0130	0.0085	0.0059	0.0043	0.0033
	4	3	0.8563	0.1889	0.0754	0.0389	0.0233	0.0153	0.0107	0.0079	0.0061
	4	4	1.3481	0.3048	0.1237	0.0646	0.0389	0.0257	0.0181	0.0134	0.0103
	5	1	0.2307	0.0473	0.0182	0.0092	0.0055	0.0036	0.0025	0.0018	0.0014
	5	2	0.4828	0.1045	0.0414	0.0213	0.0127	0.0083	0.0058	0.0043	0.0033
	5	3	0.8036	0.1802	0.0728	0.0379	0.0228	0.0150	0.0106	0.0078	0.0060
	5	4	1.2672	0.2910	0.1195	0.0629	0.0381	0.0253	0.0179	0.0133	0.0102
	5	5	2.0440	0.4779	0.1990	0.1058	0.0645	0.0431	0.0306	0.0227	0.0175
	6	1	0.2168	0.0450	0.0175	0.0090	0.0053	0.0035	0.0024	0.0018	0.0014
	6	2	0.4539	0.0995	0.0398	0.0206	0.0124	0.0082	0.0057	0.0042	0.0032
	6	3	0.7563	0.1717	0.0701	0.0368	0.0222	0.0147	0.0104	0.0077	0.0059
	6	4	1.1942	0.2776	0.1151	0.0611	0.0372	0.0248	0.0176	0.0131	0.0101
	6	5	1.9304	0.4565	0.1919	0.1028	0.0631	0.0422	0.0301	0.0224	0.0173
	6	6	3.5031	0.8400	0.3574	0.1932	0.1194	0.0804	0.0576	0.0431	0.0333
	7	1	0.2025	0.0425	0.0167	0.0086	0.0051	0.0034	0.0024	0.0017	0.0013
	7	2	0.4241	0.0939	0.0379	0.0198	0.0120	0.0079	0.0056	0.0041	0.0032
	7	3	0.7072	0.1622	0.0668	0.0353	0.0215	0.0143	0.0102	0.0076	0.0058
	7	4	1.1180	0.2624	0.1099	0.0587	0.0360	0.0241	0.0172	0.0128	0.0099
	7	5	1.8105	0.4323	0.1834	0.0989	0.0611	0.0411	0.0294	0.0220	0.0170
	7	6	3.2972	0.7976	0.3422	0.1863	0.1158	0.0784	0.0563	0.0422	0.0328
	7	7	8.8932	2.1772	0.9453	0.5198	0.3258	0.2220	0.1603	0.1209	0.0942
8	1	1	0.2628	0.0462	0.0164	0.0079	0.0045	0.0029	0.0020	0.0014	0.0011
	2	1	0.2349	0.0436	0.0158	0.0077	0.0045	0.0029	0.0020	0.0014	0.0011
	2	2	0.4812	0.0944	0.0352	0.0175	0.0102	0.0066	0.0046	0.0033	0.0025
	3	1	0.2152	0.0414	0.0153	0.0076	0.0044	0.0028	0.0019	0.0014	0.0011
	3	2	0.4414	0.0897	0.0341	0.0171	0.0100	0.0065	0.0045	0.0033	0.0025
	3	3	0.7120	0.1501	0.0582	0.0295	0.0174	0.0114	0.0079	0.0058	0.0044
	4	1	0.2003	0.0395	0.0148	0.0074	0.0043	0.0028	0.0019	0.0014	0.0011
	4	2	0.4111	0.0856	0.0330	0.0167	0.0098	0.0064	0.0045	0.0033	0.0025
	4	3	0.6639	0.1434	0.0564	0.0289	0.0172	0.0112	0.0079	0.0058	0.0044
	4	4	0.9993	0.2214	0.0886	0.0458	0.0274	0.0180	0.0127	0.0093	0.0071
	5	1	0.1882	0.0378	0.0144	0.0072	0.0042	0.0027	0.0019	0.0014	0.0010
	5	2	0.3865	0.0819	0.0320	0.0163	0.0097	0.0063	0.0044	0.0032	0.0025
	5	3	0.6248	0.1373	0.0547	0.0282	0.0168	0.0111	0.0078	0.0057	0.0044
	5	4	0.9415	0.2122	0.0859	0.0448	0.0269	0.0178	0.0125	0.0092	0.0071
	5	5	1.4085	0.3234	0.1327	0.0698	0.0422	0.0280	0.0198	0.0147	0.0113
	6	1	0.1778	0.0362	0.0139	0.0070	0.0053	0.0027	0.0019	0.0014	0.0010
	6	2	0.3654	0.0785	0.0310	0.0159	0.0095	0.0062	0.0044	0.0032	0.0024
	6	3	0.5910	0.1316	0.0530	0.0275	0.0165	0.0109	0.0077	0.0057	0.0043
	6	4	0.8914	0.2035	0.0832	0.0437	0.0264	0.0175	0.0124	0.0091	0.0070
	6	5	1.3354	0.3104	0.1286	0.0681	0.0414	0.0276	0.0196	0.0145	0.0112
	6	6	2.0883	0.4924	0.2064	0.1103	0.0675	0.0451	0.0321	0.0239	0.0185
	7	1	0.1682	0.0346	0.0134	0.0068	0.0041	0.0026	0.0018	0.0014	0.0010
	7	2	0.3457	0.0751	0.0299	0.0155	0.0093	0.0061	0.0043	0.0032	0.0024
	7	3	0.5595	0.1259	0.0511	0.0267	0.0161	0.0107	0.0075	0.0056	0.0043
	7	4	0.8446	0.1948	0.0803	0.0424	0.0258	0.0172	0.0122	0.0090	0.0069
	7	5	1.2667	0.2973	0.1243	0.0662	0.0405	0.0271	0.0192	0.0143	0.0110
	7	6	1.9844	0.4723	0.1996	0.1073	0.0660	0.0443	0.0316	0.0236	0.0182
	7	7	3.5283	0.8495	0.3628	0.1967	0.1218	0.0822	0.0589	0.0441	0.0342
	8	1	0.1579	0.0328	0.0128	0.0066	0.0039	0.0026	0.0018	0.0013	0.0010
	8	2	0.3247	0.0711	0.0286	0.0149	0.0090	0.0059	0.0042	0.0031	0.0024
	8	3	0.5257	0.1194	0.0489	0.0258	0.0156	0.0104	0.0074	0.0055	0.0042
	8	4	0.7942	0.1848	0.0769	0.0409	0.0250	0.0167	0.0119	0.0088	0.0068
	8	5	1.1923	0.2823	0.1190	0.0639	0.0393	0.0264	0.0188	0.0140	0.0109
	8	6	1.8709	0.4490	0.1913	0.1035	0.0640	0.0432	0.0309	0.0232	0.0179
	8	7	3.3369	0.8096	0.3483	0.1900	0.1183	0.0802	0.0576	0.0433	0.0336
	8	8	8.8813	1.1775	0.9468	0.5213	0.3271	0.2231	0.1612	0.1216	0.0948

where $\boldsymbol{\alpha'} = (\alpha_{1:n}, \alpha_{2:n}, \ldots, \alpha_{n:n})$ is the moment vector with $\alpha_{i:n} = E(X_{i:n})$, $\boldsymbol{\beta} = (\beta_{i,j:n}), 1 \leq i \leq j \leq n$ is the variance-covariance matrix, $\mathbf{1}$ is n-dimensional column unit vector, $\mathbf{Y} = (Y_{1:1}, \ldots, Y_{n:n})^T$ denotes the column vector of the observed order statistics from $LI(\mu, \sigma, \theta)$ and

$$\Delta = (\boldsymbol{\alpha^T \beta^{-1} \alpha})(\mathbf{1^T} \boldsymbol{\beta^{-1}} \mathbf{1}) - (\boldsymbol{\alpha^T \beta^{-1}} \mathbf{1})^{\mathbf{2}}.$$

Furthermore, the variances and covariances of these BLUEs are given by

$$\begin{aligned} Var(\hat{\mu}_{BLU}) &= \left(\boldsymbol{\alpha^T \beta^{-1} \alpha}\right)\frac{\sigma^2}{\Delta} = V_1\,\sigma^2, \\ Var(\hat{\sigma}_{BLU}) &= \left(\mathbf{1^T}\boldsymbol{\beta^{-1}}\mathbf{1}\right)\frac{\sigma^2}{\Delta} = V_2\,\sigma^2, \\ Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}) &= -\left(\boldsymbol{\alpha^T \beta^{-1}}\mathbf{1}\right)\frac{\sigma^2}{\Delta} = V_3\,\sigma^2. \end{aligned}$$

Mann (1969) points out in the course of a broader treatment that minimum mean squared error invariant estimators (BLIEs) of μ and σ are obtainable from the BLUEs $\hat{\mu}$ and $\hat{\sigma}$ as,

$$\hat{\mu}_{BLI} = \hat{\mu}_{BLU} - \frac{V_3}{1+V_2}\hat{\sigma}_{BLU}, \qquad \hat{\sigma}_{BLI} = \frac{\hat{\sigma}_{BLU}}{1+V_2},$$

respectively, where $V_1 = \frac{1}{\sigma^2}Var(\hat{\mu}_{BLU})$, $V_2 = \frac{1}{\sigma^2}Var(\hat{\sigma}_{BLU})$ and $V_3 = \frac{1}{\sigma^2}Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU})$. Furthermore the variances of these BLIEs are given by (see Arnold et al. (1998), p. 143)

$$Var(\hat{\mu}_{BLI}) = \sigma^2\left(V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}\right), \qquad Var(\hat{\sigma}_{BLI}) = \frac{\sigma^2 V_2}{(1+V_2)^2}.$$

Gupta (1952) proposed simplified linear estimators of the location and scale parameters of the normal distribution. The proposed estimators are quite useful when the problems of non-availability of the variance-covariance matrix or computational difficulties are exist. For getting quick linear estimators, we replace the variance-covariance matrix by its identity matrix. Precisely, we just replace $\boldsymbol{\beta}$ by $\boldsymbol{I}$ in (8) and (9), respectively. The so obtained estimators are LSEs or modified BLUEs of μ and σ. In a consequence of that, the LSEs of μ and σ(see Arnold et al. (2008), p.180) can be expressed as

$$\begin{aligned} \hat{\mu}_{LSE} &= \left[\frac{\boldsymbol{\alpha'\alpha}\mathbf{1'} - n\bar{\alpha}\boldsymbol{\alpha'}}{n\sum_{i=1}^{n}(\alpha_{i:n}-\bar{\alpha})^2}\right]\boldsymbol{Y} = \sum_{i=1}^{n} c_i Y_{i:n}, \\ \hat{\sigma}_{LSE} &= \left[\frac{\boldsymbol{\alpha'} - \bar{\alpha}\mathbf{1'}}{\sum_{i=1}^{n}(\alpha_{i:n}-\bar{\alpha})^2}\right]\boldsymbol{Y} = \sum_{i=1}^{n} d_i Y_{i:n}, \end{aligned}$$

where $\bar{\alpha}$ is the average of $\alpha_{i:n},\ i = 1, \ldots, n$. Further, we have

$$\begin{aligned} Var(\hat{\mu}_{LSE}) &= \left[\frac{\sum_{i=1}^{n}\alpha_{i:n}^2}{n\sum_{i=1}^{n}(\alpha_{i:n}-\bar{\alpha})^2}\right]\sigma^2 = V_4\,\sigma^2, \\ Var(\hat{\sigma}_{LSE}) &= \left[\frac{1}{\sum_{i=1}^{n}(\alpha_{i:n}-\bar{\alpha})^2}\right]\sigma^2 = V_5\,\sigma^2, \end{aligned}$$

$$Cov(\hat{\mu}_{LSE}, \hat{\sigma}_{LSE}) = \left[\frac{\sum_{i=1}^{n} \alpha_{i:n}}{n\sum_{i=1}^{n}(\alpha_{i:n}-\bar{\alpha})^2}\right]\sigma^2 = V_6\,\sigma^2.$$

The coefficients a_i's, b_i's, c_i's, d_i's ($1 \leq i \leq n$) are computed and presented in Tables 4-7, respectively. The values of $V_1 - V_6$, are presented in Tables 8 and 9. From Tables 8 and 9, we observe that the variances of the BLUEs and LSEs are decreasing with respect to n.

Table 4: Coefficients for the BLUEs of μ.

n	r	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1	1.6538	1.5909	1.5605	1.5434	1.5328	1.5256	1.5206	1.5169	1.5138
2	2	-0.6538	-0.5909	-0.5605	-0.5434	-0.5328	-0.5256	-0.5206	-0.5169	-0.5138
3	1	1.4119	1.3735	1.3527	1.3488	1.3444	1.3416	1.3388	1.3364	1.3380
3	2	-0.1587	-0.1558	-0.1565	-0.1575	-0.1594	-0.1604	-0.1604	-0.1594	-0.1639
3	3	-0.2531	-0.2176	-0.2006	-0.1912	-0.1849	-0.1811	-0.1784	-0.1769	-0.1740
4	1	1.2891	1.2637	1.2586	1.2547	1.2518	1.2539	1.2455	1.2500	1.2431
4	2	-0.0463	-0.0602	-0.0667	-0.0709	-0.0727	-0.0791	-0.0689	-0.0770	-0.0673
4	3	-0.1025	-0.0908	-0.0867	-0.0846	-0.0835	-0.0825	-0.0852	-0.0836	-0.0872
4	4	-0.1402	-0.1162	-0.1051	-0.0990	-0.0955	-0.0923	-0.0913	-0.0893	-0.0885
5	1	1.2147	1.2080	1.2027	1.2010	1.1785	1.2023	1.1985	1.2081	1.2067
5	2	-0.0061	-0.0271	-0.0356	-0.0400	-0.0415	-0.0515	-0.0424	-0.0615	-0.0663
5	3	-0.0461	-0.0543	-0.0471	-0.0474	-0.0397	-0.0412	-0.0467	-0.0442	-0.0337
5	4	-0.0716	-0.0517	-0.0548	-0.0526	-0.0177	-0.0530	-0.0506	-0.0467	-0.0521
5	5	-0.0908	-0.0748	-0.0651	-0.0608	-0.0794	-0.0564	-0.0560	-0.0556	-0.0544
6	1	1.1654	1.1643	1.1654	1.1639	1.1608	1.1611	1.1733	1.1445	1.1684
6	2	0.0106	-0.0117	-0.0225	-0.0248	-0.0249	-0.0176	-0.0468	-0.0027	-0.0328
6	3	-0.0193	-0.0245	-0.0261	-0.0283	-0.0246	-0.0390	-0.0174	-0.0325	-0.0314
6	4	-0.0393	-0.0344	-0.0333	-0.0323	-0.0359	-0.0316	-0.0391	-0.0407	-0.0335
6	5	-0.0530	-0.0430	-0.0389	-0.0369	-0.0359	-0.0353	-0.0328	-0.0307	-0.0363
6	6	-0.0643	-0.0505	-0.0445	-0.0415	-0.0393	-0.0374	-0.0369	-0.0378	-0.0342
7	1	1.1304	1.1348	1.1374	1.1432	1.1403	1.1303	1.1503	1.1522	1.1453
7	2	0.0183	-0.0026	-0.0111	-0.0241	-0.0179	-0.0006	-0.0469	-0.0583	-0.0239
7	3	-0.0052	-0.0145	-0.0184	-0.0142	-0.0209	-0.0351	-0.0002	-0.0090	0.0240
7	4	-0.0213	-0.0207	-0.0216	-0.0248	-0.0221	-0.0137	-0.0243	-0.0247	-0.0221
7	5	-0.0327	-0.0275	-0.0249	-0.0223	-0.0260	-0.0285	-0.0292	-0.0249	-0.0244
7	6	-0.0410	-0.0321	-0.0286	-0.0279	-0.0248	-0.0247	-0.0219	-0.0274	-0.0267
7	7	-0.0483	-0.0372	-0.0325	-0.0296	-0.0283	-0.0275	-0.0274	-0.0257	-0.0239
8	1	1.1046	1.1150	1.1164	1.1178	1.1342	1.1329	1.1307	1.1257	1.1322
8	2	0.0217	-0.0008	-0.0050	-0.0026	-0.0306	-0.0399	-0.0476	-0.0171	-0.0185
8	3	0.0028	-0.0073	-0.0128	-0.0218	-0.0143	0.0049	0.0135	-0.0155	-0.0180
8	4	-0.0109	-0.0132	-0.0128	-0.0148	-0.0144	-0.0289	-0.0170	-0.0183	-0.0465
8	5	-0.0203	-0.0181	-0.0193	-0.0166	-0.0120	-0.0079	-0.0197	-0.0183	0.0101
8	6	-0.0273	-0.0218	-0.0194	-0.0184	-0.0219	-0.0227	-0.0229	-0.0180	-0.0197
8	7	-0.0328	-0.0251	-0.0221	-0.0205	-0.0199	-0.0173	-0.0159	-0.0202	-0.0193
8	8	-0.0377	-0.0285	-0.0247	-0.0228	-0.0209	-0.0213	-0.0209	-0.0182	-0.0201

Note: For each n ($n = 2, \ldots, 8$), the first, second and the third lines represent $V_1 = \frac{1}{\sigma^2}Var(\hat{\mu}_{BLU})$, $V_2 = \frac{1}{\sigma^2}Var(\hat{\sigma}_{BLU})$ and $V_3 = \frac{1}{\sigma^2}Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU})$, respectively.

Note: For each n ($n = 2, \ldots, 8$), the first, second and the third lines represent $V_4 = \frac{1}{\sigma^2}Var(\hat{\mu}_{LSE})$, $V_5 = \frac{1}{\sigma^2}Var(\hat{\sigma}_{LSE})$ and $V_6 = \frac{1}{\sigma^2}Cov(\hat{\mu}_{LSE}, \hat{\sigma}_{BLU})$, respectively.

Table 5: Coefficients for the BLUEs of σ.

n	r	$\theta=0.5$	$\theta=1$	$\theta=1.5$	$\theta=2$	$\theta=2.5$	$\theta=3$	$\theta=3.5$	$\theta=4$	$\theta=4.5$
2	1	-0.3461	-0.7272	-1.1363	-1.5651	-2.0084	-2.4618	-2.9231	-3.3898	-3.8610
2	2	0.3461	0.7272	1.1363	1.5651	2.0084	2.4618	2.9231	3.3898	3.8610
3	1	-0.3195	-0.6808	-1.0763	-1.4960	-1.9350	-2.3852	-2.8441	-3.3104	-3.7884
3	2	0.1412	0.3063	0.4928	0.6944	0.9107	1.1322	1.3599	1.5914	1.8372
3	3	0.1783	0.3744	0.5835	0.8015	1.0243	1.2530	1.4842	1.7189	1.9511
4	1	-0.3037	-0.6542	-1.0429	-1.4596	-1.8940	-2.3489	-2.7927	-3.2739	-3.7304
4	2	0.0796	0.1805	0.2998	0.4335	0.5746	0.7301	0.8594	1.0368	1.1693
4	3	0.1043	0.2210	0.3489	0.4851	0.6273	0.7749	0.9328	1.0815	1.2475
4	4	0.1197	0.2525	0.3941	0.5409	0.6920	0.8438	1.0004	1.1558	1.3135
5	1	-0.2929	-0.6450	-1.0282	-1.4383	-1.6458	-2.3308	-2.7692	-3.2759	-3.7619
5	2	0.0510	0.1242	0.2177	0.3121	0.3722	0.5525	0.6326	0.8070	0.9682
5	3	0.0702	0.1746	0.2448	0.3441	0.3938	0.5425	0.6733	0.7825	0.8524
5	4	0.0817	0.1516	0.2681	0.3731	0.1832	0.5982	0.7064	0.8105	0.9500
5	5	0.0899	0.1944	0.2975	0.4088	0.7059	0.6376	0.7568	0.8757	0.9912
6	1	-0.2851	-0.6239	-1.0077	-1.4202	-1.8462	-2.3081	-2.7716	-3.1720	-3.7179
6	2	0.0351	0.0897	0.1619	0.2413	0.3175	0.4121	0.5380	0.5298	0.7238
6	3	0.0507	0.1118	0.1819	0.2615	0.3417	0.4467	0.4873	0.6122	0.6941
6	4	0.0605	0.1283	0.2033	0.2831	0.3746	0.4560	0.5721	0.6686	0.7406
6	5	0.0668	0.1412	0.2211	0.3051	0.3920	0.4818	0.5676	0.6566	0.7678
6	6	0.0718	0.1528	0.2392	0.3291	0.4202	0.5113	0.6064	0.7044	0.7914
7	1	-0.2790	-0.6141	-0.9942	-1.4125	-1.8399	-2.2755	-2.7842	-3.2689	-3.7328
7	2	0.0251	0.0687	0.1254	0.2025	0.2572	0.3002	0.4980	0.6370	0.6109
7	3	0.0384	0.0871	0.1463	0.2063	0.2894	0.3970	0.3714	0.4053	0.6178
7	4	0.468	0.1003	0.1611	0.2300	0.2961	0.3472	0.4563	0.5289	0.5708
7	5	0.0524	0.1110	0.1741	0.2402	0.3167	0.3984	0.4759	0.5459	0.6292
7	6	0.0563	0.1193	0.1870	0.2586	0.3290	0.4035	0.4735	0.5669	0.6461
7	7	0.0597	0.1275	0.2001	0.2748	0.3513	0.4289	0.5089	0.5846	0.6578
8	1	-0.2741	-0.6074	-0.9827	-1.3913	-1.8510	-2.3041	-2.7366	-3.2196	-3.7075
8	2	0.0184	0.0561	0.1001	0.1465	0.2424	0.3439	0.3972	0.4381	0.4793
8	3	0.0299	0.0698	0.1207	0.1868	0.2482	0.2534	0.3172	0.3978	0.5112
8	4	0.0375	0.0810	0.1308	0.1874	0.2403	0.3412	0.3546	0.4700	0.6457
8	5	0.0424	0.0904	0.1446	0.1999	0.2540	0.3002	0.4101	0.4686	0.3906
8	6	0.0459	0.0972	0.1523	0.2105	0.2790	0.3497	0.4079	0.4604	0.5547
8	7	0.0486	0.1033	0.1621	0.2234	0.2857	0.3451	0.4115	0.4854	0.5527
8	8	0.0510	0.1093	0.1718	0.2366	0.3010	0.3703	0.4378	0.4990	0.5731

It should be mentioned here that based on the linear estimators, we can construct confidence intervals (CIs) for the location and scale parameters. For example, based on the BLUEs of μ and σ, we can construct the CIs for μ and σ through pivotal quantities given by

$$R_1 = \frac{\hat{\mu}_{BLU} - \mu}{\hat{\sigma}_{BLU}\sqrt{V_1}}, \qquad R_2 = \frac{\hat{\sigma}_{BLU} - \sigma}{\hat{\sigma}_{BLU}\sqrt{V_2}}.$$

Constructing such CIs requires the $(1-\alpha)100$th percentiles $(0 < \alpha < 1)$ of R_1 and R_2 which can be calculated by using the BLUEs $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ via Monte Carlo simulation method. The simulated percentiles of R_1 and R_2 based on 10000 runs are obtained for different choices of n and θ and presented in Tables 10 and 11. Therefore,

Table 6: Coefficients for the LSEs of μ.

n	r	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1	1.6538	1.5909	1.5605	1.5434	1.5328	1.5256	1.5206	1.5169	1.5138
2	2	-0.6538	-0.5909	-0.5605	-0.5434	-0.5328	-0.5256	-0.5206	-0.5169	-0.5138
3	1	1.0213	0.9761	0.9532	0.9399	0.9315	0.9259	0.9217	0.9187	0.9165
3	2	0.4624	0.4653	0.4679	0.4698	0.4710	0.4720	0.4727	0.4732	0.4735
3	3	-0.4838	-0.4415	-0.4211	-0.4097	-0.4026	-0.3980	-0.3944	-0.3920	-0.3901
4	1	0.7344	0.6988	0.6806	0.6699	0.6631	0.6585	0.6553	0.6527	0.6510
4	2	0.4886	0.4801	0.4760	0.4737	0.4722	0.4711	0.4704	0.4699	0.4694
4	3	0.1700	0.1818	0.1887	0.1931	0.1960	0.1981	0.1995	0.2007	0.2015
4	4	-0.3932	-0.3608	-0.3454	-0.3368	-0.3314	-0.3278	-0.3253	-0.3234	-0.3221
5	1	0.5719	0.5425	0.5273	0.5184	0.5822	0.5088	0.5061	0.5041	0.5025
5	2	0.4339	0.4222	0.4162	0.4126	0.4717	0.4087	0.4075	0.4066	0.4060
5	3	0.2732	0.2747	0.2760	0.2769	0.3285	0.2780	0.2782	0.2786	0.2787
5	4	0.0565	0.0695	0.0769	0.0816	0.1206	0.0868	0.0884	0.0896	0.0905
5	5	-0.3356	-0.3091	-0.2966	-0.2896	-0.5032	-0.2824	-0.2804	-0.2790	-0.2778
6	1	0.4677	0.4425	0.4296	0.4219	0.4171	0.4137	0.4114	0.4097	0.4083
6	2	0.3792	0.3668	0.3602	0.3563	0.3537	0.3519	0.3506	0.3497	0.3490
6	3	0.2818	0.2879	0.2776	0.2768	0.2763	0.2760	0.2758	0.2755	0.2754
6	4	0.1642	0.1692	0.1724	0.1745	0.1758	0.1768	0.1775	0.1780	0.1785
6	5	0.0022	0.0150	0.0220	0.0265	0.0294	0.0314	0.0329	0.0340	0.0349
6	6	-0.2952	-0.2726	-0.2620	-0.2561	-0.2525	-0.2500	-0.2484	-0.2471	-0.2462
7	1	0.3953	0.3733	0.3619	0.3553	0.3510	0.3481	0.3460	0.3445	0.3434
7	2	0.3336	0.3213	0.3147	0.3108	0.3082	0.3064	0.3051	0.3041	0.3033
7	3	0.2680	0.2630	0.2604	0.2588	0.2578	0.2570	0.2565	0.2561	0.2558
7	4	0.1934	0.1944	0.1952	0.1958	0.1961	0.1964	0.1966	0.1967	01968
7	5	0.1014	0.1079	0.1117	0.1142	0.1158	0.1170	0.1178	0.1185	0.1190
7	6	-0.0269	-0.0148	-0.0082	-0.0041	-0.0014	0.0005	0.0019	0.0029	0.0037
7	7	-0.2649	-0.2452	-0.2360	-0.2309	-0.2277	-0.2256	-0.2242	-0.2231	-0.2223
8	1	0.3421	0.3226	0.3125	0.3066	0.3028	0.3002	0.2984	0.2970	0.2960
8	2	0.2966	0.2847	0.2783	0.2745	0.2719	0.2702	0.2690	0.2680	0.2673
8	3	0.2492	0.2432	0.2399	0.2380	0.2366	0.2357	0.2350	0.2345	0.2341
8	4	0.1974	0.1961	0.1955	0.1952	0.1950	0.1949	0.1947	0.1947	0.1946
8	5	0.1372	0.1401	0.1420	0.1432	0.1440	0.1446	0.1450	0.1453	0.1456
8	6	0.0621	0.0691	0.0731	0.0757	0.0774	0.0787	0.0796	0.0803	0.0808
8	7	-0.0436	-0.0322	-0.0261	-0.0224	-0.0199	-0.0181	-0.0168	-0.0158	-0.0151
8	8	-0.2413	-0.2236	-0.2154	-0.2109	-0.2082	-0.2063	-0.2050	-0.2041	-0.2034

a $100(1-\alpha)\%$ CI for μ through the pivotal quantity R_1 is found to be

$$\left(\hat{\mu}_{BLU} - \hat{\sigma}_{BLU}\sqrt{V_1}\ R_1(1-\alpha/2),\ \hat{\mu}_{BLU} - \hat{\sigma}_{BLU}\sqrt{V_1}\ R_1(\alpha/2)\right), \tag{10}$$

where $R_1(\gamma)$ is the γ100th simulated percentile of R_1. Similarly, a $100(1-\alpha)\%$ CI for σ is given by

$$\left(\frac{\hat{\sigma}_{BLU}}{1+\sqrt{V_2}\ R_2(1-\alpha/2)},\ \frac{\hat{\sigma}_{BLU}}{1+\sqrt{V_2}\ R_2(\alpha/2)}\right), \tag{11}$$

where $R_2(\gamma)$ is the γ100th simulated percentile of R_2. Based on the BLIEs, we can again construct CIs for the location and scale parameters through pivotal quantities

Table 7: Coefficients for the LSEs of σ.

n	r	$\theta=0.5$	$\theta=1$	$\theta=1.5$	$\theta=2$	$\theta=2.5$	$\theta=3$	$\theta=3.5$	$\theta=4$	$\theta=4.5$
2	1	-0.3461	-0.7272	-1.1363	-1.5651	-2.0084	-2.4618	-2.9231	-3.3898	-3.8610
2	2	0.3461	0.7272	1.1363	1.5651	2.0084	2.4618	2.9231	3.3898	3.8610
3	1	-0.2064	-0.4258	-0.6642	-0.9099	-1.1632	-1.4224	-1.8651	-1.9519	-2.2209
3	2	-0.0387	-0.0880	-0.1442	-0.2047	-0.2678	-0.3329	-0.3992	-0.4665	-0.5341
3	3	0.2451	0.5165	0.8084	1.1147	1.4311	1.7553	2.0843	2.4184	2.7550
4	1	-0.1453	-0.2992	-0.4614	-0.6300	-0.8034	-0.9805	-1.1608	-1.3426	-1.5272
4	2	-0.0716	-0.1543	-0.2422	-0.3356	-0.4321	-0.5308	-0.6314	-0.7332	-0.8358
4	3	0.0239	0.0454	0.0656	0.0853	0.1048	0.1244	0.1444	0.1641	0.1843
4	4	0.1929	0.4072	0.6380	0.8802	1.1306	1.3870	1.6478	1.9118	2.1787
5	1	-0.1115	-0.2283	-0.3507	-0.4777	-0.5936	-0.7413	-0.8768	-1.0138	-1.1522
5	2	-0.0701	-0.1481	-0.2317	-0.3189	-0.3991	-0.5009	-0.5945	-0.6890	-0.7846
5	3	-0.0219	-0.0498	-0.0815	-0.1154	-0.1471	-0.1872	-0.2241	-0.2620	-0.3000
5	4	0.0430	0.0869	0.1318	0.1775	0.2187	0.2715	0.3194	0.3679	0.4169
5	5	0.1607	0.3394	0.5321	0.7345	1.3166	1.1580	1.3760	1.5969	1.8199
6	1	-0.0903	-0.1839	-0.2817	-0.3829	-0.4869	-0.5930	-0.7009	-0.8102	-0.9202
6	2	-0.0637	-0.1334	-0.2074	-0.2844	-0.3638	-0.4447	-0.5269	-0.6103	-0.6944
6	3	-0.0345	-0.0748	-0.1189	-0.1653	-0.2133	-0.2625	-0.3125	-0.3631	-0.4142
6	4	0.0007	-0.0017	-0.0062	-0.0117	-0.0179	-0.0244	-0.0312	-0.0379	-0.0451
6	5	0.0493	0.1011	0.1549	0.2102	0.2668	0.3244	0.3828	0.4420	0.5014
6	6	0.1385	0.2928	0.4593	0.6343	0.8151	1.0003	1.1887	1.3796	1.5726
7	1	-0.0757	-0.1536	-0.2347	-0.3187	-0.4048	-0.4927	-0.5820	-0.6724	-0.7637
7	2	-0.0572	-0.1189	-0.1841	-0.2519	-0.3215	-0.3926	-0.4647	-0.5378	-0.6113
7	3	-0.0375	-0.0801	-0.1259	-0.1740	-0.2235	-0.2742	-0.3257	-0.3777	-0.4304
7	4	-0.0151	-0.0343	-0.0561	-0.0794	-0.1037	-0.1286	-0.1540	-0.1797	-0.2058
7	5	0.0124	0.0232	0.0333	0.0429	0.0524	0.0619	0.0715	0.0811	0.0907
7	6	0.0509	0.1051	0.1618	0.2205	0.2805	0.3416	0.4036	0.4663	0.5296
7	7	0.1223	0.2587	0.4059	0.5606	0.7206	0.8846	1.0513	1.2202	1.3909
8	1	-0.0651	-0.1317	-0.2009	-0.2724	-0.3458	-0.4206	-0.4967	-0.5736	-0.6512
8	2	-0.0514	-0.1064	-0.1643	-0.2242	-0.2858	-0.3486	-0.4124	-0.4770	-0.5423
8	3	-0.0372	-0.0788	-0.1232	-0.1695	-0.2171	-0.2657	-0.3151	-0.3652	-0.4156
8	4	-0.0217	-0.0474	-0.0756	-0.1054	-0.1362	-0.1677	-0.1999	-0.2324	-0.2651
8	5	-0.0036	-0.0101	-0.0182	-0.0273	-0.0370	-0.0471	-0.0573	-0.0677	-0.0784
8	6	0.0188	0.0372	0.0555	0.0738	0.0923	0.1111	0.1299	0.1490	0.1682
8	7	0.0505	0.1048	0.1619	0.2211	0.2817	0.3435	0.4062	0.4696	0.5336
8	8	0.1098	0.2324	0.3648	0.5040	0.6479	0.7954	0.9454	1.0975	1.2510

given by

$$R_3 = \frac{\hat{\mu}_{BLI} - \mu}{\hat{\sigma}_{BLI}\sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}}, \qquad R_4 = \frac{\hat{\sigma}_{BLI} - \sigma}{\sigma \frac{\sqrt{V_2}}{1+V_2}}.$$

Tables 12 and 13 presents the percentage points of R_3 and R_4 based on 10000 runs and different choices of n and θ. With the BLIEs and the use of Tables 12 and 13, we can determine a $100(1-\alpha)\%$ CI for μ through the pivotal quantity R_3 as

$$P\Bigg(\hat{\mu}_{BLI} - \hat{\sigma}_{BLI}\sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}}R_3(1-\alpha/2) \leq \mu$$

Table 8: Variances and covariances of the BLUEs of μ and σ in terms of σ^2.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	5.7765	1.1539	0.4428	0.2249	0.1334	0.0873	0.0612	0.0451	0.0345
	0.8106	0.8512	0.8823	0.9054	0.9233	0.9357	0.9458	0.9537	0.9600
	-1.6509	-0.7314	-0.4537	-0.3247	-0.2513	-0.2039	-0.1713	-0.1474	-0.1293
3	2.1909	0.4205	0.1573	0.0787	0.0460	0.0299	0.0208	0.0152	0.0116
	0.3943	0.4176	0.4354	0.4487	0.4580	0.4654	0.4707	0.04751	0.4788
	-0.6082	-0.2590	-0.1574	-0.1114	-0.0854	-0.0690	-0.0577	-0.0496	-0.0434
4	1.2038	0.2234	0.0820	0.0404	0.0235	0.0152	0.0106	0.0077	0.0059
	0.2580	0.2750	0.2879	0.2973	0.3041	0.3093	0.3133	0.3168	0.3187
	-0.3275	-0.1351	-0.0808	-0.0566	-0.0432	-0.0348	-0.0292	-0.0251	-0.0218
5	0.7775	0.1407	0.0506	0.0248	0.0139	0.0092	0.0064	0.0046	0.0035
	0.1908	0.2075	0.2146	0.2222	0.1995	0.2314	0.2350	0.2369	0.2384
	-0.2083	-0.848	-0.0494	-0.0344	-0.0230	-0.0210	-0.0177	-0.0150	-0.0131
6	0.5503	0.0968	0.0345	0.0168	0.0097	0.0061	0.0042	0.0031	0.0022
	0.1510	0.1625	0.1710	0.1773	0.1818	0.1848	0.1875	0.1900	0.1906
	-0.1457	-0.0571	-0.0334	-0.0232	-0.0176	-0.0140	-0.0117	-0.0102	-0.0086
7	0.4133	0.0711	0.0251	0.0120	0.0069	0.0044	0.0030	0.0022	0.0016
	0.1247	0.1347	0.1421	0.1473	0.1512	0.1540	0.1562	0.1581	0.1585
	-0.1083	-0.0416	-0.0242	-0.0166	-0.0126	-0.0101	-0.0084	-0.0073	-0.0061
8	0.3234	0.0546	0.0191	0.0091	0.0051	0.0033	0.0023	0.0016	0.0012
	0.1061	0.1149	0.1215	0.1261	0.1292	0.1320	0.1340	0.1346	0.1365
	-0.0841	-0.0317	-0.0183	-0.0125	-0.0094	-0.0076	-0.0064	-0.0052	-0.0048

Table 9: Variances and covariances of the LSEs of μ and σ in terms of σ^2.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	3.1627	2.8801	2.7496	2.6775	2.6335	2.6037	2.5832	2.5683	2.5559
	0.2396	1.0578	2.5826	4.8996	8.0676	12.1213	17.0892	22.9819	29.8147
	-0.7988	-1.5867	-2.4103	-3.2663	-4.1487	-5.0497	-5.9667	-6.8945	-7.8293
3	1.4911	1.3643	1.3050	1.2721	1.2517	1.2386	1.2286	1.2218	1.2165
	0.1042	0.4582	1.1155	2.1125	3.4730	5.2155	7.3436	9.8765	12.8082
	-0.3473	-0.6873	-1.0411	-1.4083	-1.7859	-2.1729	-2.5641	-2.9623	-3.3634
4	0.9619	0.8822	0.8449	0.8240	0.8111	0.8023	0.7964	0.7918	0.7886
	0.0640	0.2810	0.6829	1.2917	2.1216	3.1827	4.4822	6.0223	7.8118
	-0.2135	-0.4214	-0.6374	-0.8610	-1.0911	-1.3259	-1.5651	-1.8064	-2.0513
5	0.7060	0.6485	0.6215	0.6063	0.6730	0.5906	0.5862	0.5830	0.5804
	0.0455	0.1993	0.4839	0.9144	1.4648	2.2504	3.1681	4.2569	5.5193
	-0.1518	-0.2990	-0.4516	-0.6096	-0.8324	-0.9375	-1.1061	-1.2769	-1.4491
6	0.5561	0.5114	0.4903	0.4784	0.4710	0.4660	0.4626	0.4600	0.4581
	0.0350	0.1532	0.3716	0.7016	1.1511	1.7250	2.4273	3.2612	4.2273
	-0.1168	-0.2298	-0.3468	-0.4677	-0.5919	-0.7186	-0.8475	-0.9782	-1.1100
7	0.4580	0.4216	0.4043	0.3946	0.3885	0.3844	0.3815	0.3794	0.3778
	0.0283	0.1238	0.3001	0.5664	0.9289	1.3918	1.9580	2.6294	3.4082
	-0.0945	-0.1858	-0.2801	-0.3776	-0.4777	-0.5798	-0.6836	-0.7886	-0.8949
8	0.3890	0.3582	0.3436	0.3354	0.3302	0.3267	0.3243	0.3225	0.3211
	0.0237	0.1036	0.2510	0.4734	0.7761	1.1625	1.6352	2.1960	2.8456
	-0.0792	-0.1555	-0.2342	-0.3156	-0.3991	-0.4843	-0.5709	-0.6586	-0.7471

$$\leq \hat{\mu}_{BLI} - \hat{\sigma}_{BLI} \sqrt{V_1 - \frac{V_3^2(2+V_2)}{(1+V_2)^2}} R_3(\alpha/2) \Bigg) = 1 - \alpha, \qquad (12)$$

Similarly, we can determine a $100(1-\alpha)\%$ CI for σ, through the pivotal quantity R_4 as

$$P\left(\frac{\hat{\sigma}_{BLI}}{1 + \frac{\sqrt{V_2}}{1+V_2} R_4(1-\alpha/2)} \leq \sigma \leq \frac{\hat{\sigma}_{BLI}}{1 + \frac{\sqrt{V_2}}{1+V_2} R_4(\alpha/2)} \right) = 1 - \alpha. \qquad (13)$$

4 Linear predictors

The prediction problems do arise naturally in many real-life applications of ordered data.

Let $Y_{1:n}, Y_{2:n}, \ldots, Y_{r:n}$ be the first r observed order statistics from LI(μ, σ, θ) with pdf given(6). Our aim here is to predict the future kth order statistics, $Y_{k:n}$ based on the first r order statistics, where $r < k \leq n$. This type of prediction is called one-sample prediction problem which is based on Type-II censoring sample. For further studies, see for example, Raqab (1997) and Kaminsky and Nelson (1998). Suppose a machine consists of n components and fails whenever k of these components fail. Our observations only consist of the first r failure times, and the aim is to predict the failure time of that machine. In location-scale families of distributions, the most well-known predictor is the best linear unbiased predictor (BLUP). Its respective explicit form is

$$\hat{Y}_{BLUP} = \hat{Y}_{k:n} = \hat{\mu}_{BLU} + \hat{\sigma}_{BLU}\ \alpha_{k:n} + \boldsymbol{\omega}^T \boldsymbol{\beta}^{-1}(\boldsymbol{Y} - \hat{\mu}_{BLU}\mathbf{1} - \hat{\sigma}_{BLU}\boldsymbol{\alpha}),$$

where

$$\boldsymbol{\omega}^T = \left(Cov(Y_{1:n}, Y_{k:n}), \ldots, Cov(Y_{r:n}, Y_{k:n})\right).$$

The best linear invariant predictor (BLIP) of Y can be obtained based on the BLUP of Y as follows:

$$\hat{Y}_{BLIP} = \hat{Y}_{BLUP} - \left(\frac{V_4}{1+V_2}\right)\hat{\sigma}_{BLU},$$

where $\hat{Y}_{BLUP}$ is the BLUP of $Y_{k:n}$ and

$$V_4 = (1 - \boldsymbol{\omega}'\boldsymbol{\beta}^{-1}\mathbf{1})V_3 + (\alpha_{k:n} - \boldsymbol{\omega}'\boldsymbol{\beta}^{-1}\boldsymbol{\alpha})V_2.$$

5 Illustrative examples and comparisons

5.1 Illustrative examples

In this section, based on two data sets we explain how the different estimators proposed here work in practice. Further, a Monte Carlo simulation is performed to evaluate the efficiencies of the linear estimators developed in the previous sections.

Example 1 (simulated data): For given values of $\theta = 0.5$, $\mu = 0$ and $\sigma = 1$, we have generated $n = 8$ order statistics values from the LI distribution as follows:

0.5851, 0.7607, 0.9755, 2.9328, 3.2378, 3.8766, 6.1355, 6.4767.

Table 10: Simulated percentage points of R_1 and R_2.

		R_1				R_2			
θ	n	2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
0.5	2	-0.7529	-0.7333	13.0622	27.7782	-1.0749	-1.0395	1.9811	2.6091
	3	-0.8822	-0.8506	5.0296	7.9833	-1.3543	-1.2529	1.9045	2.4410
	4	-0.9413	-0.9036	3.6679	5.4652	-1.4825	-1.3391	1.8683	2.3416
	5	-0.9761	-0.9345	3.2057	4.5485	-1.5531	-1.3851	1.8310	2.2882
	6	-0.9975	-0.9540	2.9480	4.1477	-1.6154	-1.4307	1.8152	2.2707
	7	-1.0124	-0.9674	2.7241	3.7862	-1.6304	-1.4389	1.8061	2.2348
	8	-1.0244	-0.9759	2.6579	3.6683	-1.6786	-1.4656	1.7797	2.2045
1	2	-0.7338	-0.7117	13.0521	27.4121	-1.0523	-1.0209	19862	2.6890
	3	-0.8451	-0.8202	4.9017	8.0291	-1.3317	-1.2312	1.8900	2.4212
	4	-0.8947	-0.8667	3.5695	5.4201	-1.4552	-1.3221	1.8930	2.4070
	5	-0.9199	-0.8905	3.1622	4.5983	-1.5262	-1.3653	1.8300	2.3112
	6	-0.9401	-0.9086	2.88572	4.0713	-1.5722	-1.3992	1.8262	2.3020
	7	-0.9495	-0.9181	2.7132	3.8052	-1.6182	-1.4301	1.8072	2.2370
	8	-0.9581	-0.9257	2.5990	3.6222	-1.6571	-1.4545	1.8001	2.2520
1.5	2	-0.7213	-0.7011	12.6521	26.9102	-1.0340	-1.0032	1.9941	2.6290
	3	-0.8289	-0.8053	4.8680	7.7731	-1.3112	-1.2170	1.9192	2.4703
	4	-0.8747	-0.8502	3.5642	5.3482	-1.4420	-1.3080	1.8862	2.3740
	5	-0.9017	-0.8765	3.1260	4.4855	-1.5160	-1.3611	1.8440	2.3170
	6	-0.9183	-0.8925	2.8710	4.1032	-1.5777	-1.4050	1.8342	2.3027
	7	-0.9276	-0.9011	2.6501	3.7333	-1.5960	-1.4142	1.8260	2.2641
	8	-0.9357	-0.9085	2.5992	3.6333	-1.6380	-1.4432	1.8032	2.2370
2	2	-0.7119	-0.6919	12.8801	26.9822	-1.0220	-0.9926	1.9911	2.6450
	3	-0.8183	-0.7952	4.8430	7.9672	-1.2970	-1.2062	1.9040	2.4463
	4	-0.8660	-0.8412	3.5410	5.4132	-1.4240	-1.3002	1.9102	2.4280
	5	-0.8904	-0.8659	3.1311	4.5712	-1.5112	-1.3530	1.8522	2.3470
	6	-0.9080	-0.8831	2.8180	4.0512	-1.5460	-1.3802	1.8403	2.3175
	7	-0.9212	-0.8954	2.6970	3.8011	-1.5953	-1.4143	1.8162	2.2588
	8	-0.9294	-0.9045	2.5798	3.6217	-1.6332	-1.4382	1.8142	2.2667
2.5	2	-0.7074	-0.6881	12.5632	26.7376	-1.0122	-0.9832	1.9985	2.6409
	3	-0.8140	-0.7913	4.8697	7.7863	-1.2864	-1.1977	1.9296	2.4927
	4	-0.8608	-0.8372	3.5475	5.3458	-1.4195	-1.2912	1.8951	2.3942
	5	-0.9036	-0.8799	3.1731	4.5571	-1.6001	-1.4466	1.0328	2.5697
	6	-0.9003	-0.8756	2.8538	4.0847	-1.5568	-1.3918	1.8420	2.3188
	7	-0.9177	-0.8931	2.6456	3.7506	-1.5775	-1.4024	1.8349	2.2775
	8	-0.9368	-0.9106	2.6222	3.6834	-1.6198	-1.4323	1.8141	2.2537
3	2	-0.7040	-0.6840	12.6673	26.6613	-1.0068	-0.9788	2.0105	2.6882
	3	-0.8105	-0.7883	4.9053	7.8884	-1.2787	-1.1934	1.9569	2.5301
	4	-0.8572	-0.8331	3.6425	5.4890	-1.4151	-1.2930	1.8766	2.4121
	5	-0.8840	-0.8609	3.1161	4.5712	-1.4915	-1.3502	1.8757	2.3470
	6	-0.9079	-0.8831	2.8760	4.0701	-1.5411	-1.3769	1.8646	2.3457
	7	-0.9186	-0.8934	2.7342	3.8654	-1.5807	-1.4040	1.8321	2.2883
	8	-0.9511	-0.9247	2.5189	3.5472	-1.6131	-1.4242	1.8132	2.2649
3.5	2	-0.7004	-0.6812	12.8191	26.8717	-1.0011	-0.9731	1.9909	2.6683
	3	-0.8083	-0.7858	4.8524	7.9472	-1.2723	-1.1870	1.9102	2.4697
	4	-0.8531	-0.8290	3.5312	5.4037	-1.4018	-1.2814	1.9182	2.4395
	5	-0.8805	-0.8566	3.1358	4.5878	-1.4880	-1.3363	1.8608	2.3656
	6	-0.9074	-0.8828	2.8465	4.1008	-1.5283	-1.3653	1.8496	2.3350
	7	-0.9219	-0.8946	2.7068	3.8373	-1.5755	-1.3994	1.8205	2.2703
	8	-0.9873	-0.9607	2.4228	3.4386	-1.6138	-1.4229	1.8161	2.2753
4	2	-0.6995	-0.6808	12.5228	26.6943	-0.9965	-0.9691	1.9993	2.6512
	3	-0.8080	-0.7857	4.8770	7.8158	-1.2688	-1.1828	1.9314	2.5056
	4	-0.8542	-0.8308	3.5510	5.3525	-1.4003	-1.2771	1.8998	1.4057
	5	-0.8868	-0.8634	3.1263	4.5186	-1.4779	-1.3341	1.8587	2.3475
	6	-0.9029	-0.8767	2.8652	4.1201	-1.5386	-1.3799	1.8454	2.3331
	7	-0.9612	-0.9353	2.5501	3.6465	-1.5623	-1.3919	1.8402	2.2924
	8	-0.9418	-0.9167	2.6524	3.7283	-1.6072	-1.4239	1.8251	2.2658

Table 11: Continued.

		R_1				R_2			
θ	n	2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
4.5	2	-0.6983	-0.6786	12.6577	26.6347	-0.9944	-0.9674	2.0095	2.6981
	3	-0.8069	-0.7850	4.9213	7.9226	-1.2642	-1.1819	1.9631	2.5403
	4	-0.8507	-0.8265	3.6408	5.4906	-1.4017	-1.2820	1.8815	2.4193
	5	-0.8853	-0.8612	3.1329	4.6077	-1.4790	-1.3420	1.8801	2.3585
	6	-0.9317	-0.9068	2.9570	4.2005	-1.5304	-1.3691	1.8716	2.3574
	7	-0.9791	-0.9547	2.7484	3.9100	-1.5714	-1.3978	1.8414	2.3028
	8	-1.0702	-1.0381	2.4479	3.4968	-1.5994	-1.4167	1.8186	2.2733

Table 12: Simulated percentage points of R_3 and R_4.

		R_3				R_4			
θ	n	2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
0.5	2	-0.7356	-0.6850	26.53	55.71	-1.975	-1.942	1.076	1.705
	3	-0.8554	-0.8140	7.350	11.99	-1.981	-1.883	1.262	1.829
	4	-0.9126	-0.8598	4.947	6.920	-1.995	-1.835	1.303	1.774
	5	-0.9375	-0.8919	4.074	5.655	-1.986	-1.823	1.394	1.844
	6	-0.9655	-0.9127	3.708	5.094	-1.981	-1.812	1.425	1.889
	7	-0.5295	-0.4553	1.271	1.580	-0.3792	-0.0172	6.880	7.918
	8	-1.000	-0.9486	2.966	3.984	-1.986	-1.776	1.487	1.884
1	2	-0.7308	-0.6910	27.68	57.04	-1.976	-1.946	1.072	1.815
	3	-0.8364	-0.8020	7.646	11.75	-1.976	-1.874	1.302	1.822
	4	-0.8777	-0.8388	4.956	7.537	-1.982	-1.850	1.319	1.832
	5	-0.8965	-0.8521	4.3547	6.2412	-1.993	-1.840	1.398	1.833
	6	-0.9213	-0.8799	3.650	5.098	-2.008	-1.834	1.421	1.834
	7	-0.9344	-0.8941	3.263	4.525	-1.982	-1.790	1.441	1.867
	8	-0.9443	-0.9058	2.982	4.213	-1.999	-1.803	1.456	1.862
1.5	2	-0..7230	-0.6786	24.63	54.40	-1.973	-1.944	1.073	1.726
	3	-0.8282	-0.7908	7.668	12.30	-1.969	-1.884	1.312	1.885
	4	-0..8648	-0.8319	5.078	7.352	-1.980	-1.839	1.353	1.870
	5	-0.8881	-0.8566	3.901	5.595	-1.964	-1.817	1.444	1.899
	6	-0.9022	-0.8701	3.611	5.033	-1.984	-1.817	1.420	1.871
	7	-0.9152	-0.8846	3.285	4.575	-1.999	-1.824	1.398	1.862
	8	-0.9236	-0.8901	2.985	4.124	-1.975	-1.785	1.466	1.884
2	2	-0.7232	-0.6835	26.53	53.81	-1.972	-1.939	1.087	1.713
	3	-0.8190	-0.7853	8.000	12.76	-1.980	-1.885	1.218	1.715
	4	-0.8617	-0.8321	5.046	7.593	-1.971	-1.846	1.319	1.836
	5	-0.8783	-0.8444	4.089	5.943	-1.990	-1.831	1.324	1.794
	6	-0.8990	-0.8712	3.470	5.129	-1.984	-1.810	1.411	1.890
	7	-0.9135	-0.8852	3.323	4.844	-1.998	-1.820	1.433	1.899
	8	-0.9211	-0.8939	3.030	4.179	-2.000	-1.792	1.472	1.952
2.5	2	-0.7215	-06825	25.97	53.41	-1.973	-1.944	1.006	1.673
	3	-0.8245	-0.7941	7.732	11.65	-1.950	-1.865	1.294	1.899
	4	-0.8623	-0.8334	5.209	7.614	-1.967	-1.844	1.367	1.860
	5	-0.9036	-0.8746	4.112	5.967	-2.063	-1.898	1.631	2.224
	6	-0.8982	-0.8693	3.545	4.918	-1.968	-1.806	1.401	1.869
	7	-0.9129	-0.8791	3.280	4.588	-1.997	-1.827	1.436	1.877
	8	-0.9308	-0.9029	3.100	4.327	-1.964	-1.791	1.444	1.906
3	2	-0.7201	-0.6822	26.55	55.86	-1.973	-1.945	1.004	1.666
	3	-0..8204	-0.7865	7.689	11.68	-1.969	-1.884	1.250	1.758
	4	-0.8582	-0.8220	4.876	7.410	-1.974	-1.853	1.302	1.812
	5	-0.8802	-0.8546	4.240	5.843	-1.985	-1.833	1.410	1.868
	6	-0.9058	-0.8749	3.552	5.042	-1.975	-1.812	1.416	1.863
	7	-0.9184	-0.8898	3.309	4.700	-1.983	-1.796	1.440	1.874
	8	-0.9269	-0.8948	3.183	4.347	-1.985	-1.792	1.463	1.921

Table 13: Continued.

		R_3				R_4			
θ	n	2.5%	5%	95%	97.5%	2.5%	5%	95%	97.5%
3.5	2	-0.7210	-0.6834	25.95	54.38	-1.972	-1.942	.9838	1.615
	3	-0.8199	-0.7844	7.919	12.35	-1.957	-1.877	1.305	1.888
	4	-0.8560	-0.8272	5.098	7.282	-1.969	-1.852	1.292	1.798
	5	-0.8759	-0.8489	4.036	5.685	-1.948	-1.804	1.362	1.842
	6	-0.9080	-0.8800	3.699	5.229	-1.962	-1.799	1.441	1.941
	7	-0.9213	-0.8920	3.318	4.644	-1.975	-1.7987	1.436	1.875
	8	-0.9260	-0.8933	3.075	4.444	-1.965	-1.771	1.479	1.888
4	2	-0.7210	-0.6834	24.34	49.89	-1.970	-1.943	.9974	1.736
	3	-0.4809	-0.4559	5.713	9.184	-4.238	-3.963	5.964	7.576
	4	-0.8586	-0.8278	5.498	7.802	-1.968	-1.849	1.340	1.878
	5	-0.8802	-0.8401	1.627	1.881	-1.587	-1.396	2.679	3.248
	6	-0.9022	-0.8682	3.650	5.079	-1.968	-1.801	1.434	1.905
	7	-0.9663	-0.9374	3.184	4.445	-1.952	-1.794	1.443	1.907
	8	-0.9466	-0.9164	3.108	4.322	-1.964	-1.783	1.469	1.928
4.5	2	-0.7216	-0.6845	26.82	56.91	-1.970	-1.944	.9675	1.736
	3	-0.8230	-0.7895	7.968	13.15	-1.962	-1.872	1.221	1.787
	4	-0.8584	-0.8289	5.240	7.554	-1.961	-1.836	1.396	1.899
	5	-0.8915	-0.8622	4.108	6.258	-1.972	-1.835	1.378	1.886
	6	-0.9382	-0.9076	3.851	5.511	-1.961	-1.799	1.4417	1.909
	7	-0.8280	-0.7795	3.834	5.145	-1.994	-1.814	1.442	1.942
	8	-0.7837	-0.7552	2.763	3.826	-2.146	-2.376	2.337	2.651

For this data, we obtain the MLEs of μ and σ as

$$\hat{\mu}_{MLE} = 0.1614, \qquad \hat{\sigma}_{MLE} = 0.8877.$$

Based on these data and from Tables 4 and 5, we have determined the BLUEs of μ and σ to be

$$\begin{aligned}
\hat{\mu}_{BLU} &= a_1Y_{1:n} + a_2Y_{2:n} + a_3Y_{3:n} + a_4Y_{4:n} + a_5Y_{5:n} + a_6Y_{6:n} + a_7Y_{7:n} + a_8Y_{8:n} \\
&= (1.1046 \times 0.5851) + (0.0217 \times 0.7607) + (0.0028 \times 0.9755) \\
&\quad -(0.0109 \times 2.9328) - (0.0203 \times 3.2378) - (0.0273 \times 3.8766) \\
&\quad -(0.0328 \times 6.1355) - (0.0377 \times 6.4767) = 0.0166, \\
\hat{\sigma}_{BLU} &= b_1Y_{1:n} + b_2Y_{2:n} + b_3Y_{3:n} + b_4Y_{4:n} + b_5Y_{5:n} + b_6Y_{6:n} + b_7Y_{7:n} + b_8Y_{8:n} \\
&= (-0.2741 \times 0.5851) + (0.0184 \times 0.7607) + (0.0299 \times 0.9755) \\
&\quad +(0.0375 \times 2.9328) + (0.0424 \times 3.2378) + (0.0459 \times 3.8766) \\
&\quad +(0.0486 \times 6.1355) + (0.0510 \times 6.4767) = 0.9364.
\end{aligned}$$

The corresponding variances and covariances of $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ (see Table 8) are computed to be

$$Var(\hat{\mu}_{BLU}) = 0.3234\sigma^2, \quad Var(\hat{\sigma}_{BLU}) = 0.1061\sigma^2, \quad Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}) = -0.0841\sigma^2.$$

The BLIEs and the corresponding variances are

$$\hat{\mu}_{BLI} = 0.0878, \quad \hat{\sigma}_{BLI} = 0.8466, \quad Var(\hat{\mu}_{BLI}) = 0.3176, \quad Var(\hat{\sigma}_{BLI}) = 0.0867.$$

From Tables 6 and 7, we then derive the LSEs of μ and σ as

$$\begin{aligned}
\hat{\mu}_{LSE} &= c_1Y_{1:n} + c_2Y_{2:n} + c_3Y_{3:n} + c_4Y_{4:n} + c_5Y_{5:n} + c_6Y_{6:n} + c_7Y_{7:n} + c_8Y_{8:n} \\
&= (0.3421 \times 0.5851) + (0.2966 \times 0.7607) + (0.2492 \times 0.9755) \\
&\quad +(0.1974 \times 2.9328) + (0.1372 \times 3.2378) + (0.0621 \times 3.8766) \\
&\quad -(0.0436 \times 6.1355) - (0.2413 \times 6.4767) = 0.1024, \\
\hat{\sigma}_{LSE} &= d_1Y_{1:n} + d_2Y_{2:n} + d_3Y_{3:n} + d_4Y_{4:n} + d_5Y_{5:n} + d_6Y_{6:n} + d_7Y_{7:n} + d_8Y_{8:n} \\
&= (-0.0651 \times 0.5851) - (0.0514 \times 0.7607) - (0.0372 \times 0.9755) \\
&\quad -(0.0217 \times 2.9328) - (0.0036 \times 3.2378) + (0.0188 \times 3.8766) \\
&\quad +(0.0505 \times 6.1355) + (0.1098 \times 6.4767) = 0.9050.
\end{aligned}$$

Similarly, the corresponding variances and covariances of $\hat{\mu}_{LSE}$ and $\hat{\sigma}_{LSE}$ (see Table 9) are computed to be:

$$Var(\hat{\mu}_{LSE}) = 0.3890\sigma^2, \quad Var(\hat{\sigma}_{LSE}) = 0.0237\sigma^2, \quad Cov(\hat{\mu}_{LSE}, \hat{\sigma}_{LSE}) = -0.0792\sigma^2.$$

From Tables 10 and 11, it follows from (10) and (11), the 95% CIs for μ and σ based on R_1 and R_2 are (-1.9365, 0.5620) and (0.5450, 2.0661), respectively. Also, From Tables 12 and 13, it follows from (12) and (13), the 95% CIs for μ and σ based on R_3 and R_4 are (-1.8132, 0.5649) and (0.6023, 2.2558), respectively.

Suppose that we want to find the BLUP of $Y_{7:8}$ based on the first six observed order statistic. From Table 1, we have $\alpha_{7:8} = 5.4626$, when $\theta = 0.5$. From Tables 2-3, for $\theta = 0.5$, we have

$$\begin{aligned}
\boldsymbol{\omega}^T = &(Cov(Y_{1:8}, Y_{7:8}), Cov(Y_{2:8}, Y_{7:8}), Cov(Y_{3:8}, Y_{7:8}), Cov(Y_{4:8}, Y_{7:8}), Cov(Y_{5:8}, Y_{7:8}), \\
&Cov(Y_{6:8}, Y_{7:8})) = (0.1682, 0.3457, 0.5595, 0.8446, 1.2667, 1.9844).
\end{aligned}$$

Further, the vector of the first six observed order statistics is

$$\boldsymbol{Y}^T = (0.5851,\ 0.7607,\ 0.9755,\ 2.9328,\ 3.2378,\ 3.8766),$$

and the vector of standard means is (from Table 1)

$$\boldsymbol{\alpha}^T = (0.5915,\ 1.1669,\ 1.7642,\ 2.4189,\ 3.1783,\ 4.1264).$$

Also $\boldsymbol{\beta}_{6\times 6}$ is the variance-covariance matrix of the six standard order statistics which can obtained from Tables 2-3. So, the BLUP of the 7th order statistic is:

$$\hat{Y}_{7:8} = \hat{\mu}_{BLU} + \hat{\sigma}_{BLU}\ \alpha_{7:8} + \boldsymbol{\omega}^T\boldsymbol{\beta}^{-1}(\boldsymbol{Y} - \hat{\mu}_{BLU}\mathbf{1} - \hat{\sigma}_{BLU}\boldsymbol{\alpha}) = 5.1282.$$

Also, the BLIP of the 7th order statistic, $Y_{7:8}$ is $\hat{Y}_{BLIP} = 4.9447$.

Example 2 (real data): In this example, we analyze the total annual rainfall (in inches) during Jun recorded at Los Angeles Civic Center from 2017 to 2023 (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm). The data set is displayed as follows

0.19, 0.32, 1.77, 2.05, 2.44, 5.95, 8.95.

Let us first discuss how we can obtain an initial guess of the shape parameter θ from the LI distribution. Determination of a shape parameter of a location-scale family has been discussed in Tiku and Akkaya (2004). Since the skewness and kurtosis measures are independent of the location and scale parameters, we can identify a plausible value of the shape parameter by matching (approximately) the sample skewness and kurtosis with their corresponding distributional values. In our case, an initial guess value of θ can be determined by equating the sample skewness with the population skewness. The skewness ($\sqrt{\beta_1}$) of the LI distribution is given by (Ghitany et al., 2008)

$$\sqrt{\beta_1} = \frac{2(\theta^3 + 6\theta^2 + 6\theta + 2)}{(\theta^2 + 4\theta + 2)^{\frac{3}{2}}}.$$

The values of LI skewness are computed for different values of θ and reported in Table 14. For the above data set, the sample skewness is found to be 1.6637. Based on this value and Table 14, the plausible value of θ can be chosen to be $\theta = 1.5$. Now, when $\theta =$

Table 14: LI skewness for different values of θ.

θ	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
$\sqrt{\beta_1}$	1.51	1.61	1.69	1.75	1.79	1.83	1.85	1.87	1.89

1.5, the maximum likelihood estimates (MLEs) of the location and scale parameters are computed as $\hat{\mu}_{MLE} = 0.19$ and $\hat{\sigma}_{MLE} = 3.2034$. Therefore, Kolmogorov- Smirnov (K-S) statistic of the distance between the fitted and the empirical distribution functions is 0.2225 and the corresponding p-value is 0.8296. Therefore, it is reasonable to assume that the LI distribution is appropriate fitting distribution of the above data set. The order statistics for above data set are as follows

$$0.19, \quad 0.32, \quad 1.77, \quad 2.05, \quad 2.44, \quad 5.95, \quad 8.95.$$

From Tables 4 and 5, it follow that the BLUEs of μ and σ are

$$\begin{aligned}
\hat{\mu}_{BLU} &= a_1 Y_{1:n} + a_2 Y_{2:n} + a_3 Y_{3:n} + a_4 Y_{4:n} + a_5 Y_{5:n} + a_6 Y_{6:n} + a_7 Y_{7:n} \\
&= (1.1374 \times 0.19) - (0.0111 \times 0.32) - (0.0184 \times 1.77) - (0.0216 \times 2.05) \\
&\quad -(0.0249 \times 2.44) - (0.0286 \times 5.95) - (0.0325 \times 8.95) = -0.3861, \\
\hat{\sigma}_{BLU} &= b_1 Y_{1:n} + b_2 Y_{2:n} + b_3 Y_{3:n} + b_4 Y_{4:n} + b_5 Y_{5:n} + b_6 Y_{6:n} + b_7 Y_{7:n} \\
&= (-0.9942 \times 0.19) + (0.1254 \times 0.32) + (0.1463 \times 1.77) + (0.1611 \times 2.05) \\
&\quad +(0.1741 \times 2.44) + (0.1870 \times 5.95) + (0.2001 \times 8.95) = 3.7688.
\end{aligned}$$

The BLIEs of the location and scale parameters are given by $\hat{\mu}_{BLI} = -0.3062$ and $\hat{\sigma}_{BLI} = 3.2999$. From Tables 6 and 7, the LSEs of μ and σ are

$$\begin{aligned}
\hat{\mu}_{LSE} &= c_1 Y_{1:n} + c_2 Y_{2:n} + c_3 Y_{3:n} + c_4 Y_{4:n} + c_5 Y_{5:n} + c_6 Y_{6:n} + c_7 Y_{7:n} \\
&= (0.3619 \times 0.19) + (0.3147 \times 0.32) + (0.2604 \times 1.77) + (0.1952 \times 2.05) \\
&\quad +(0.1117 \times 2.44) - (0.0082 \times 5.95) - (0.2360 \times 8.95) = -0.7488, \\
\hat{\sigma}_{LSE} &= d_1 Y_{1:n} + d_2 Y_{2:n} + d_3 Y_{3:n} + d_4 Y_{4:n} + d_5 Y_{5:n} + d_6 Y_{6:n} + d_7 Y_{7:n}
\end{aligned}$$

$$
\begin{aligned}
&= (-0.2347 \times 0.19) - (0.1841 \times 0.32) - (0.1259 \times 1.77) - (0.0561 \times 2.05) \\
&\quad +(0.0333 \times 2.44) + (0.1618 \times 5.95) + (0.4059 \times 8.95) = 4.2354.
\end{aligned}
$$

The corresponding variances and covariances of $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ (see Table 8) are computed to be

$$Var(\hat{\mu}_{BLU}) = 0.0251\sigma^2, \quad Var(\hat{\sigma}_{BLU}) = 0.1421\sigma^2, \quad Cov(\hat{\mu}_{BLU}, \hat{\sigma}_{BLU}) = -0.0242\sigma^2.$$

The variances of $\hat{\mu}_{BLI}$ and $\hat{\sigma}_{BLI}$ are

$$Var(\hat{\mu}_{BLI}) = 0.0246\sigma^2, \qquad Var(\hat{\sigma}_{BLI}) = 0.1089\sigma^2.$$

Similarly, the corresponding variances and covariances of $\hat{\mu}_{LSE}$ and $\hat{\sigma}_{LSE}$ (see Table 9) are computed to be

$$Var(\hat{\mu}_{LSE}) = 0.4043\sigma^2, \quad Var(\hat{\sigma}_{LSE}) = 0.3001\sigma^2, \quad Cov(\hat{\mu}_{LSE}, \hat{\sigma}_{LSE}) = -0.2801\sigma^2.$$

From Tables 10 and 11, it follows from (10) and (11), the 95% CIs for μ and σ based on R_1 and R_2 are (-2.6152, 0.1677) and (2.0334, 9.4605), respectively. Also, From Tables 12 and 13, it follows from (12) and (13), the 95% CIs for μ and σ based on R_3 and R_4 are (-2.6766, 0.1679) and (2.3238, 11.078), respectively.

Suppose that we want to find the BLUP of $\hat{Y}_{6:7}$ based on the first five order statistics. From Table 1, we have $\alpha_{6:7} = 1.4726$, when $\theta = 1.5$. From Tables 2-3, we have

$$
\begin{aligned}
\boldsymbol{\omega}^T &= (Cov(Y_{1:7}, Y_{6:7}), Cov(Y_{2:7}, Y_{6:7}), Cov(Y_{3:7}, Y_{6:7}), Cov(Y_{4:7}, Y_{6:7}), Cov(Y_{5:7}, Y_{6:7})) \\
&= (0.0175, 0.0398, 0.0701, 0.1151, 0.1919).
\end{aligned}
$$

Further, the vector of the first five observed order statistics is

$$\boldsymbol{Y}^T = (0.19,\ 0.32,\ 1.77,\ 2.05,\ 2.44)$$

and the vector of standard means is (from Table 1)

$$\boldsymbol{\alpha}^T = (0.1511,\ 0.3197,\ 0.5136,\ 0.7463,\ 1.0443).$$

Also $\boldsymbol{\beta}_{5\times 5}$ is the variance-covariance matrix of the first five standard order statistics which can obtained from Tables 2-3. So, the BLUP of the 6th order statistic is:

$$\hat{Y}_{6:7} = \hat{\mu}_{BLU} + \hat{\sigma}_{BLU}\ \alpha_{6:7} + \boldsymbol{\omega}^T \boldsymbol{\beta}^{-1}(\boldsymbol{Y} - \hat{\mu}_{BLU}\mathbf{1} - \hat{\sigma}_{BLU}\boldsymbol{\alpha}) = 4.8163.$$

Also, the BLIP of the 6th order statistic, $Y_{6:7}$ is $\hat{Y}_{BLIP} = 4.5154$.

5.2 Comparisons

In this section, we carry out a Monte Carlo simulation study to evaluate the performances of the different estimators of μ and σ. In this comparative study, we have randomly generated 1000 ordered sample $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ from the LI distribution with pdf given in (1). We then obtained the BLUEs, LSEs and MLEs of μ and σ for different values of n and θ. We then compared the performances of these estimators in terms of the estimated relative efficiency (ERE) criterion. We obtained the ERE of the BLUEs with respect to the LSEs and the MLEs. For notational convenience, let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimates of θ. The ERE of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is defined by

$$ERE(\hat{\theta}_1, \hat{\theta}_2) = \frac{\sum_{i=1}^{M}(\hat{\theta}_2(i) - \theta)^2}{\sum_{i=1}^{M}(\hat{\theta}_1(i) - \theta)^2},$$

where $\hat{\theta}_k(i)(k = 1, 2)$ are estimates of θ at the ith iteration over M runs. Based on this definition, we can compute ERE($\hat{\mu}_{BLU}, \hat{\mu}_{MLE}$) and ERE($\hat{\sigma}_{BLU}, \hat{\sigma}_{MLE}$). Clearly, ERE($\hat{\mu}_{BLU}, \hat{\mu}_{LSE}$) and ERE($\hat{\sigma}_{BLU}, \hat{\sigma}_{LSE}$) can be computed using the variances of the BLUEs and LSEs presented in Section 3.

We present the results of the EREs of the BLUEs to the MLEs as well as the EREs of the BLUEs to the LSEs for $n = 1, 2, \ldots, 8$ and $\theta = 0.5(0.5)4.5$. The results are displayed in Tables 15-18, respectively. From Tables 15 and 16, it is observed that the BLUEs of μ work better than the MLEs of μ but for estimating the scale parameter σ, the MLEs work better than the BLUEs.

Table 15: EREs of $\hat{\mu}_{BLU}$ with respect to $\hat{\mu}_{MLE}$.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1.0457	1.2158	1.1926	1.0303	1.0063	1.0180	1.0384	1.0255	1.0501
3	1.3781	1.3573	1.2995	1.3098	1.3675	1.3321	1.3651	1.3674	1.2796
4	1.4157	1.6041	1.5737	1.5061	1.5426	1.4462	1.4261	1.4704	1.6002
5	1.7729	1.5758	1.6574	1.6238	1.5720	1.6210	1.5610	1.6170	1.6760
6	1.8078	1.8138	1.6943	1.7250	1.7270	1.6652	1.6102	1.7158	1.7180
7	1.9024	1.8086	1.8308	1.7192	1.7245	1.6372	1.7129	1.7775	1.7055
8	1.8706	1.8238	1.8310	1.7479	1.6966	1.6774	1.7401	1.8203	1.7277

Table 16: EREs of $\hat{\sigma}_{BLU}$ with respect to $\hat{\sigma}_{MLE}$.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	0.7470	0.7662	0.6466	0.5863	0.5207	0.5038	0.5128	0.4769	0.5414
3	0.7556	0.6705	0.7154	0.6536	0.6652	0.6577	0.7041	0.7039	0.6476
4	0.7251	0.8306	0.7986	0.7754	0.7716	0.7310	0.7573	0.7795	0.7342
5	0.8490	0.8056	0.8813	0.8415	0.7420	0.8516	0.7672	0.8313	0.8272
6	0.9062	0.8683	0.8371	0.8797	0.8532	0.8610	0.8384	0.8211	0.8108
7	0.9202	0.9095	0.9136	0.8211	0.9298	0.8241	0.8840	0.8664	0.8153
8	0.9642	0.8907	0.8926	0.9544	0.8676	0.9142	0.8508	0.8997	0.8757

From Tables 17 and 18 for both μ and σ, we observe that the BLUEs work better than the LSEs.

Now, let us compare the BLUEs and BLIEs using the relative efficiency criterion (REC). Since the mean squared errors (MSEs) of BLUEs are equal to their corresponding variances, we have

$$MSE(\hat{\mu}_{BLU}) = \sigma^2 V_1, \qquad MSE(\hat{\sigma}_{BLU}) = \sigma^2 V_2.$$

On the other hand, the MSEs of BLIEs of μ and σ can be obtained as

$$MSE(\hat{\mu}_{BLI}) = \sigma^2 \left(V_1 - \frac{V_3^2}{1 + V_2} \right), \qquad MSE(\hat{\sigma}_{BLI}) = \frac{\sigma^2 V_2}{1 + V_2}.$$

Therefore, we can readily obtain the RECs of the BLIEs of μ and σ with respect to

Table 17: The EREs of $\hat{\mu}_{BLU}$ with respect to $\hat{\mu}_{LSE}$.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1	1	1	1	1	1	1	1	1
3	1.2946	1.4182	1.4628	1.3810	1.4733	1.3730	1.4720	1.6007	1.4908
4	1.6648	1.6228	1.6893	1.7425	1.9306	1.9228	1.9816	1.8569	1.9947
5	1.7848	2.0850	1.9948	2.3234	2.0014	2.4457	2.2762	2.3622	2.5839
6	2.0356	2.2442	2.6259	2.7546	2.6738	2.6723	2.8457	2.8541	3.2809
7	2.2744	2.7816	2.8026	2.9951	3.1082	3.6201	3.2945	3.4338	3.3677
8	2.4653	2.8403	3.3484	3.4353	3.8923	3.7011	3.9799	3.8284	3.8613

Table 18: The EREs of $\hat{\sigma}_{BLU}$ with respect to $\hat{\sigma}_{LSE}$.

n	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 2.5$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$	$\theta = 4.5$
2	1	1	1	1	1	1	1	1	1
3	1.1337	1.1520	1.1598	1.1131	1.1912	1.0596	1.1705	1.2126	1.1804
4	1.2363	1.2245	1.1673	1.2715	1.2757	1.2961	1.2280	1.2324	1.2816
5	1.2896	1.2327	1.2665	1.3577	1.3135	1.3125	1.2328	1.3302	1.3572
6	1.3109	1.2352	1.2675	1.3959	1.3291	1.3376	1.2522	1.3472	1.3594
7	1.3337	1.2612	1.3462	1.3976	1.3998	1.3756	1.3838	1.4547	1.4300
8	1.3760	1.3672	1.3787	1.4460	1.4131	1.4169	1.4010	1.4730	1.4896

their corresponding BLUEs as follows

$$\begin{aligned} REC(\hat{\mu}_{BLI}, \hat{\mu}_{BLU}) &= \frac{MSE(\hat{\mu}_{BLU})}{MSE(\hat{\mu}_{BLI})} = \frac{V_1}{V_1 - \frac{V_3^2}{1+V_2}} \geq 1, \\ REC(\hat{\sigma}_{BLI}, \hat{\sigma}_{BLU}) &= \frac{MSE(\hat{\sigma}_{BLU})}{MSE(\hat{\sigma}_{BLI})} = 1 + V_2 \geq 1. \end{aligned}$$

Therefore, both of the BLIEs of μ and σ perform better than the corresponding BLUEs in terms of MSEs. The Mathematical Package Maple 16 used to obtain the numerical results.

6 Concluding remarks

In this paper, order statistics from Lindley distribution and associated inference are considered. We also computed the means, variances, and covariances of the order statistics for various values of the shape parameter. The BLUEs, BLIEs and LSEs estimators of the location and scale parameters of the Lindley distribution were discussed. The BLUEs and BLIEs are then used to construct the confidence intervals (CIs) for the location and scale parameters. Some numerical comparisons have been conducted between the BLUEs, BLIEs, LSE and MLEs of the location and scale parameters based on MSEs. The paper is established based on the assumption that the shape parameter is known. The procedure that used in our manuscript (and in the mentioned above papers) is applicable for location-scale family of distributions and the shape parameter (if there is any) is assumed to be known. However, if θ is unknown, then $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ are no longer BLUEs. But we can consider $\hat{\mu}_{BLU}$ and $\hat{\sigma}_{BLU}$ as approximate best unbiased estimators (ABLUEs) if we replace θ with a suitable estimate, $\hat{\theta}$ say, in expressions (8) and (9).

We observe that for all considered cases, the BLUEs of μ perform better than their corresponding MLEs in the sense of mean squared error (MSE) as the $REC(\hat{\mu}_{BLU}, \hat{\mu}_{MLE}) > 1$. On the other hand, the values of $REC(\hat{\sigma}_{BLU}, \hat{\sigma}_{MLE})$ are not greater than 1, therefore the BLUE of σ performs better than the MLE of σ. We observe that for all considered cases, the $REC(\hat{\mu}_{BLI}, \hat{\mu}_{BLU})$ and $REC(\hat{\sigma}_{BLI}, \hat{\sigma}_{BLU})$ are greater than 1, therefore, both of the BLIEs of μ and σ perform better than the corresponding BLUEs in terms of MSEs. We observe that for all considered cases, the $REC(\hat{\mu}_{BLU}, \hat{\mu}_{LSE})$ and $REC(\hat{\sigma}_{BLU}, \hat{\sigma}_{LSE})$ are greater than 1, therefore, both of the BLUEs of μ and σ perform better than the corresponding LSE. Overall speaking, based on the simulation results here, we would recommend using BLIEs for the linear estimators of of μ and σ. For future work, we discuss linear inference for the lifetime distribution of components based on a Type-II censored lifetime data of reliability systems with known signatures.

References

Ali, S., Aslam, M. and Kazmi, S.M.A. (2013). A study of the effect of the loss function on Bayes estimate, posterior risk and hazard function for Lindley distribution. *Applied Mathematical Modelling*, **37**(8):426–440.

Al-Mutairi D.K., Ghitany, M.E. and Kundu, D. (2013). Inference on stress strength reliability from Lindley distribution. *Communication in Statistics-Theory and Methods*, **42**(8):1443–1463.

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998). *Records*. New York: John Wiley and Sons.

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (2008). *A First Course in Order Statistics*. Philadelphia: Classic Edition, SIAM.

Asgharzadeh, A., Bakouch, H.S., Nadarajah, S. and Sharafi, F. (2016). A new weighted Lindley distribution with application, *Brazilian Journal of Probability and Statistics*.

Balakrishnan, N. and Clifford Cohen, A. (1990). *Order Statistics and Inference*. Elsevier.

Bakouch, H.S., Al-Zahrani, B.M., Al-Shomrani, A.A., Marchi, V.A.A. and Louzada, F. (2012). An extended Lindley distribution. *Journal of the Korean Statistical Society*, **41**:75–85.

Chesneau, C., Tomy, L. and Gillariose, J. (2021). A New Modified Lindley Distribution with Properties and Applications. *Journal of Statistics Management and System*, **24**(7):1383–1403.

Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics and Computers in Simulation*, **78**(4):493–506.

Gomez-Deniz, E., Sordo, M.A. and Calderin-Ojeda, E. (2014). The log-Lindley distribution as an alternative to the beta regression model with applications in insurance. *Insurance: Mathematics and Economics*, **54**:49–57.

Gupta, A. (1952). Estimation of the mean and standard deviation of a normal population from a censored sample, *Biometrika*, **39**(3/4):260-273.

Gupta, P.K. and Singh, B. (2013). Parameter estimation of Lindley distribution with hybrid censored data. *International Journal of System Assurance Engineering and Management*, **4**(4):378–385.

Kaminsky, K.S. and Nelson, P.I. (1998). Prediction of order statistics. *Handbook of Statistics*, **17**:431–450.

Kharazmi, O., Kumar, K. and Sanku Dey. (2023). Power modified Lindley distribution: Properties, classical and Bayesian estimation and regression model with applications. *Austrian Journal of Statistics*, **52**(3):71–95.

Krishna, H. and Kumar, K. (2011). Reliability estimation in Lindley distribution with progressively type II right censored sample. *Mathematics and Computers in Simulation*, **82**(2):281–294.

Kumar, D., Nassar, M., Malik, M.R. and Dey, S. (2023). Estimation of the location and scale parameters of generalized Pareto distribution based on progressively type-II censored order statistics, *Annals of Data Science*, **10**(2):349–383.

Lindley, D.V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B (Methodological)*, **20**:102–107.

Mahmoud, M.A.W., Sultan, K.S. and Amer, S.M. (2003). Order statistics frominverseweibull distribution and associated inference. *Computational statistics & data analysis*, **42**(1-2):149–163.

Mann, N.R. (1969). Optimum estimators for linear functions of location and scale parameters. *The Annals of Mathematical Statistics*, **40**(6):2149–2155.

MirMostafaee, S.M.T.K., Asgharzadeh, A. and Fallah, A. (2016. Record values from NH distribution and associated inference. *METRON*, **74**:37–59.

Nadarajah, S., Bakouch, H.S. and Tahmasbi, R. (2011). A generalized Lindley distribution. *Sankhyā, B*, **73**:331–359.

Raqab, M.Z. (1997). Modified maximum likelihood predictors of future order statistics from normal samples. *Computational Statistics & Data Analysis*, **25**(1):91–106.

Tiku, M.L. and Akkaya, A.D. (2004). *Robust Estimation and Hypothesis Testing.* New Delhi: New Age International (P) Publishers.

Zakerzadeh, H. and Dolati, A. (2009). Generalized Lindley distribution. *Journal of Mathematical Extension*, **3**:13–25.