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Research Paper

The impact of reinsurance strategies on the ruin probability in the context of dependent and heavy-tailed losses

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Abstract: The frequency and severity of extreme events have increased in recent years in many areas. In the context of risk management for insurance companies, reinsurance provides a safe solution as it offers coverage for large claims. This paper investigates the impact of dependent extreme losses on ruin probabilities under four types of reinsurance: excess of loss, quota share, largest claims, and ecomor. To achieve this, we use the dynamic GARCH-extreme value theory-copula combined model to fit the specific features of claim data and provide more accurate estimates than classical models. We derive the surplus processes and asymptotic ruin probabilities under the Cramer-Lundberg risk process. Using a numerical example with real-life data, we illustrate the effects of dependence and the behavior of reinsurance strategies for both insurers and reinsurers. This comparison includes risk premiums, surplus processes, risk measures, and ruin probabilities. The findings show that the GARCH-extreme value theory-copula model mitigates the over- and under-estimation of risk associated with extremes and lowers the ruin probability for heavy-tailed distributions.

Keywords: Copula; Expected shortfall; Extreme value theory; Reinsurance; Ruin probability; Value-at-risk.
Mathematics Subject Classification (2010): 91B05, 62P05.

1 Introduction

Risk analysis and control became more significant and open to improvements due to the globally increasing frequency and severity of catastrophic events Kratz (2019).

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examples of such events in recent history include the Covid-19 pandemic, the 2008 global financial crisis, and 21st-century earthquakes and wildfires. By disregarding extreme losses and treating them as outliers, insurance companies inadvertently expose themselves to systematic and insolvency risks. This can result in uncontrollably high premiums or even lead to the exclusion of such risks during the underwriting process. Catastrophic losses play a substantial role in the credibility of risk modeling, and as such, they are typically covered by reinsurance. A reliable risk model for controlling the ruin process cannot be established without examining the heavy tails and taking reinsurance contracts into account. These models serve as the foundation for premium and reserve calculations, as well as for making reinsurance decisions.

Extreme value theory (EVT) has gained popularity in financial literature as a suitable model for right-skewed and heavy-tailed data Embrechts et al. (1999). The preferred approach in EVT, known as the peaks over threshold (POT), utilizes the generalized Pareto distribution (GPD) to model the data exceedances over a predetermined threshold level. While the majority of EVT literature focuses on finance and has relatively fewer applications in engineering, EVT is also relevant to loss data. This is especially true in the context of insurance and reinsurance companies that cover catastrophic losses. Considering the long-term accruing nature of loss data, it exhibits non-stationarity and volatility, making it necessary to account for these characteristics in the modeling process to enhance risk measure estimations.

Previous works by Beirlant and Teugels (1992) and Watts et al. (2006) use EVT models, but they do not consider the dependency structure and time series aspects. Copulas allow the flexible incorporation of the co-movement of multi-variate data in the modeling. Jin et al. (2022) utilize the generalized autoregressive conditional heteroskedasticity-extreme value theory-copula (GARCH-EVT-copula) approach to model the financial markets and investigate their risk diversification. Frees and Valdez (1998) show how to apply copulas to dependent actuarial data and to price a reinsurance contract with retention and limit. Additionally, Chukwudum (2019) studies extremal dependence using the GPD and copulas to quantify the risk capital under excess of loss reinsurance strategy.

Portfolio stability in the insurance industry is significantly impacted by large claims, and reinsurers are particularly concerned about these extreme claims due to the substantial risk associated with reinsurance agreements. Therefore, it is necessary to conduct precise studies on the asymptotic distributions of large claims and their influence on ruin probabilities when implementing optimal reinsurance strategies. In their work, Eling et al. (2009) show the link between solvency assessment and ruin probability while considering the available investment opportunities. Eryilmaz and Gebizlioglu (2017) study finite time non-ruin probability under exchangeable and dependent claims. Weng et al. (2009) investigate the asymptotic ruin probabilities within a discrete-time risk model featuring constant interest rates and the class of regular variation. Furthermore, Konstantinides (2011) presents the distribution parameters in terms of conver-gence and explores the implications of heavy-tailed distributions. The sensitivity of the ruin probabilities concerning loading factors, means of claim frequency, and severity are considered by Chan and Yang (2005).

The necessity of measuring the validity of models and analyzing their effects is also a consideration in insurance legislation. If the primary concerns of the decisionmakers, such as reserves, expected ruin time, and ruin probabilities, are addressed by an accurate model, more reliable results can be obtained. Given the complexity involved in measuring and modeling extremes and selecting the optimal reinsurance strategy, this paper seeks to assess the effectiveness of incorporating EVT models in reinsurance pricing and quantifying ruin probabilities in case of dependent risk. To illustrate the practical utility of this approach, a real-life data application is implemented. The expected outcomes of this paper are to guide experts in reinsurance pricing and decision-making, especially when dealing with dependent extreme losses.

The rest of the paper is organized as follows. Section 2 outlines the theory behind the combined model, known as GARCH-EVT-copula. The proposed model examines the extreme values with multivariate dependence and time-varying parameters. In Section 3, we focus on the compound loss distribution and the premium estimations for both the reinsurer and insurer, considering four reinsurance strategies. We derive the related reinsurance-based surplus processes and ruin probabilities in Section 4. The application of the proposed model to real-world data is demonstrated in Section 5, where we examine the optimal reinsurance strategy for both the insurer and the reinsurer. Finally, Section 6 presents a summary of the conclusions and highlights potential areas for future research.

2 Methodology

o account for potential time-dependent trends, volatility clustering, and dependence on loss data, we integrate autoregressive moving average-GARCH (ARMA-GARCH), EVT, and copula models. Therefore, we propose a combined model composed of these three com-ponents. To ease calculations, we initially process the data by applying a logarithmic transformation and subsequently employ ARMA-GARCH to model the volatility and trend of each marginal component. Given that actuarial loss data often exhibit extremes in the right tail, we apply EVT to the standardized residuals of the ARMA-GARCH model to better capture the heavy-tail risk. Then, we utilize copulas to model the dependence structure within the data. Finally, we estimate one-step-ahead risk measures (value-at-risk (VaR) and expected shortfall (ES)) based on the obtained parameter estimations and assess the validity and accuracy using backtesting methods. This approach involves several comprehensive steps, and based on that, we provide a summary of the techniques employed.

2.1 Dynamic ARMA-GARCH

ARMA-GARCH is utilized within a mowing window framework to ensure that time-dependent changes in the data are adequately represented in the model. In this framework, model parameters are re-estimated in each mowing window of length w as the new observations become available. The length of the dynamic window, which retains the last observed w data points, should be chosen sufficiently long to fit a GPD to the exceedances, yet short enough to effectively reflect the changes in the recent past. By this, the objective is to reflect accurately not only the current risks but also the potential risks in the future.

This approach represents an improvement in accounting for volatility and mean changes in the observations while deriving an independent and identically distributed series suitable for EVT. By integrating a time series model with EVT, we also ensure that the iid observations assumption of EVT is satisfied.

The ARMA-GARCH process is commonly employed for univariate data to extract information on the trend and the volatility Glosten et al. (1993). As its simple form, an ARMA(1, 1)-GARCH(1, 1) process is shown as

$$\begin{aligned} x_{j,t} &= \mu_{j,t} + \delta_j x_{j,t-1} + \theta_j \epsilon_{j,t-1} + \epsilon_{j,t}, \\ \epsilon_{j,t} &= z_{j,t} + \sqrt{y_{j,t}}, \\ y_{j,t} &= \alpha_{0,j} + \alpha_j \epsilon_{j,t}^2 + \beta_j y_{j,t-1}, \end{aligned}$$

where $x_{j,t}$ denotes the return of asset j at time t, $\mu_{j,t}$ is the moment, $\epsilon_{j,t}$ is the residual, $delta_j$ and θ_j are the ARMA coefficients. We assume, $z_{j,t}$ to be iid random variables with mean 0 and variance 1. The variance process is denoted by $y_{j,t}$ at which, $\alpha_{0,j}$, $\alpha_j, \beta_j > 0$, and $\alpha_j + \beta_j < 1$ are required conditions for the GARCH parameters.

2.2 Generalized Pareto distribution

Recent extreme events have demonstrated the inadequacy of models based on the Gaussian distribution assumption Marimoutou et al. (2009). Insurance data typically exhibits characteristics such as right-skewness, heavy tails, and high peaks. Attempting to model such data using the traditional Gaussian distribution can be insufficient and lead to an underestimation of tail risk. Therefore, our objective is to employ EVT with the POT approach, allowing us to utilize and extract information more effectively from the tail of the distribution.

Let X be a random variable with distribution function, F_X, The Pickands-Balkema-De Haan theorem Pickands (1975) and Balkema and De Haan (1974). states that for most heavy-tailed distributions, given a sufficiently high threshold $u \geq 0$, approaching the upper limit $x_F = \sup\{x \in R : F_X(x) < 1\} \leq \infty$, the distribution of the observations over u converges to a GPD. More precisely, for the excess distribution function of X over u, defined as $F_u(x)$,

$$F_u(x) = P(X - u \leq x | X > u) = \frac{F_X(u + x) - F_X(u)}{1 - F_X(u)}, \tag{1}$$

where $0 \leq x \leq x_F - u$, we obtain the following convergence behaviour

$$\lim_{x \to x_F} \sup_{0 \leq x \leq x_F - u} |F_u(x) - G_{\kappa,\sigma}(x)| = 0, \tag{2}$$

for some $\sigma > 0$, and κ. (1) and (2) directly leads an expression to the distribution of F_X such that,

$$F_X(x) = \bar{F}_X(u) G_{\kappa,\sigma(x-u)} + F_X(u), \tag{3}$$

where $\bar{F} = 1 - F_X$. The GPD, $G_{\kappa,\sigma(x)}$, bounded from below by $u > 0$ with tail index κ and scale parameter σ, is given by

$$G_{\kappa,\sigma(x)} = \begin{cases} 1 - \left(1 + \kappa \frac{x}{\sigma}\right)^{-\frac{1}{\kappa}}, & \text{if } \kappa \neq 0, \\ 1 - \exp\left(\frac{-x}{\sigma}\right), & \text{if } \kappa = 0, \end{cases} \tag{4}$$

where $0 \leq x \leq x_F - u$ for $\kappa > 0$, and $0 \leq x \leq -\frac{\sigma}{\kappa}$ for $\kappa < 0$. The mean excess function of X, $e(u)$ is

$$e(u) = E\big[X - u|X > u\big] = \big(\frac{\sigma}{1-\kappa} + \frac{\kappa}{1-\kappa}u\big), \qquad 0 \leq x \leq x_F,$$

where $0 < \kappa < 1$. The graph of the mean excess function is the most commonly used tool to determine the proper threshold. The appropriate threshold u is selected from the mean excess plot where it becomes roughly linear Charpentier and Flachaire (2021). Then, maximum likelihood estimator can be used for the parameters of $\{\kappa, \sigma\}$. The log-likelihood function of (4), in a sample with length n becomes

$$l(\kappa, \sigma|x) = \log L(\kappa, \sigma|x) = -n \log \sigma - \big(1 + \frac{1}{\kappa}\big) \sum_{i=1}^{n} \log\big(1 + \kappa \frac{x_i}{\sigma}\big),$$

for $\kappa \neq 0$. The parameter estimators can be obtained with respect to κ and σ as solutions of the following equations

$$\frac{1}{\kappa^2} \sum_{i=1}^{n} \log\big(1 + \kappa \frac{x_i}{\sigma}\big) - \sum_{i=1}^{n} \frac{x_i}{\sigma + \kappa x_i} = 0,$$
$$-n + (1+\kappa) \sum_{i=1}^{n} \frac{x_i}{\sigma + \kappa x_i} = 0,$$

respectively, which holds for $\kappa > -0.5$ (Smith, 1985).

VaR is the most frequently used risk measure for the financial and insurance industry and is a benchmark for reserve estimations since the establishment of the Basel framework. VaR is defined as the q-quantile of the distribution,

$$VaR_q(X) = \inf\big\{x \in R : F_X(x) \geq q\big\} = F_X^{-1}(q). \tag{5}$$

where $F_X^{-1}(q)$ represents the quantile function of the distribution. As useful as it is, VaR does not give any information beyond the q-quantile and it is not a coherent risk measure. On the other hand, contrary to VaR, ES is a coherent risk measure and expressed as

$$\begin{aligned} ES_q(X) &= E\big[X|X > VaR_q(X)\big] \\ &= VaR_q(X) + E\big[X - VaR_q(X)|X > VaR_q(X)\big] \\ &= VaR_q(X) + e\big(VaR_q(X)\big). \end{aligned} \tag{6}$$

The semi-parametric estimators of VaR_q and ES_q, following (3) where F_u follows a GPD, are given as

$$V\hat{a}R_q(X) = u + \frac{\hat{\sigma}}{\hat{\kappa}}\big[\big(\frac{n}{N_u}(1-q)\big)^{-\hat{\kappa}} - 1\big], \tag{7}$$

$$\hat{ES}_q(X) = \frac{V\hat{a}R_q(X)}{1-\hat{\kappa}} + \frac{\hat{\sigma} - \hat{\kappa}}{1-\hat{\kappa}}, \tag{8}$$

respectively McNeil (1999). Here, N_u denotes the number of exceedances above the threshold, $N_u = \{1 \leq i \leq n : x_i > u\}$, and $\frac{N_u}{n}$ is the empirical estimator of $P(X > u)$. Similarly, following from (3), $\bar{F}_X(x)$ can be approximated by

$$\bar{F}_X(x) = \frac{N_u}{n}\big[1 + \hat{\kappa}\frac{(x-u)}{\hat{\sigma}}\big]^{-\frac{1}{\hat{\kappa}}}. \tag{9}$$

2.3 Dependence

It is well-known that many risks may exhibit multidimensional dependence, especially the interdependence in the tail distributions which can result in misleading modeling of catastrophic and systemic risks Frees and Wang (2005). To assess the dependence in the tails, we employ tail dependence coefficients, which define the probability of one variable taking an extreme value at the same level another variable takes Ledford and Tawn (1996). Given two random variables, (X, Y), with marginal distribution functions F_X and F_Y, the upper and lower tail dependence coefficients λ_u and λ_t respectively, are defined as

$$\begin{aligned}\lambda_u &= \lim_{q\to 0} P\big(X > F_X^{-1}(q)|Y > F_Y^{-1}(q)\big),\\ \lambda_t &= \lim_{q\to 1} P\big(X < F_X^{-1}(q)|Y\big) < F_Y^{-1}(q)\big),\end{aligned}$$

when the limits exist. The coefficients λ_u and λ_t can also be used to fit a suitable copula to the data pairs of (X, Y).

To benefit from traditional linear dependence measures, like the Pearson correlation coefficient, certain conditions must be met. Hence, we employ copulas as a more flexible and suitable approach to identify and support capital adequacy decisions based on risk measures. While ARMA-GARCH allows for the modeling of marginal distributions, copulas incorporate the dynamic dependence structure into the model residuals. This process also provides an iid data basis, satisfying the assumptions required for applying EVT.

Assume that random variables X_i, where $i = 1, \dots, d$, have marginal distribution functions $F_{X_i}(x_i) = u_i$, and they are linked by their multivariate distribution function F. A d-dimensional copula, $C(u) = \{u_1, \dots, u_d\}$, is a multivariate joint distribution function defined as

$$F(x_1, \dots, x_d) = F\big(F_{X_1}^{-1}(u_1), \dots, F_{X_n}^{-1}(u_n)\big),$$

by Sklar (1985). If F_{X_i} is continuous then C is unique, which provides a practical way to work with d-dimensional distributions.

There are mainly two families of copulas: Elliptical ones with symmetrical tail dependencies and Archimedean ones with asymmetrical tail dependencies. The varying characteristics of observations can be incorporated into the model by dynamically implementing a copula and time series model Patton (2006). We choose the best-fitting copula in the application section from a selection that includes Gaussian, Student's-t, Clayton, Frank, Gumbel, Joe, Plackett, Galambos, Husler Reiss, Tawn and their rotated variations to assist in modeling the dependence structure of the distribution's tail.

3 Reinsurance under EVT

The effects of the combined GARCH-EVT-copula model on the probability of ruin under the reinsurance strategies mostly remain unknown Jorion (2007). The ruin probability highly depends on the large claims in the tail and intends to provide a basis for premiums, reserves, capital requirements, and obtain estimations by standards such as Solvency II and IFRS17 Tsai and Chen (2011). Following from Mikosch (1997), we assume that the claim number at time t, $N(t)$, follows a homogeneous Poisson counting process, with intensity $E[N(t)] = \lambda t > 0$, such that

$$P\big(N(t) = k\big) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \qquad k \in N.$$

Independent from the claim number, let $X_i, i = 1, \dots, N(t)$, be the non-negative claim amount random variable with mean μ_X , variance, σ_X, and common distribution function F. Thus, the corresponding total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$, for $N(t) > 0$, constitutes a compound Poisson process.

Various reinsurance strategies can be applied based on different risk structures. For example, quota share rein-surance is generally used in small claims branches. Excess of loss can be considered an extreme value reinsurance contract, as it covers claims above a predetermined limit. Similarly, the largest claims reinsurance and ecomor provide coverage only for the pre-defined number of upper tail claims.

The total claim amount $S(t)$ is divided between $R(t)$ for the reinsurer and $I(t)$ for the insurer, as $S(t) = R(t) + I(t)$. The total premium, $c(t)$, is subdivided between $P_R(t)$ for the reinsurer and $P_I(t)$ for the insurer, as $c(t) = P_R(t) + P_I(t)$. We follow an expected value premium principle with loading factors ρ for the insurer and ν for the reinsurer to cover the loss of each party,

$$\begin{aligned} P_R(t) &= (1+\nu)E[R(t)] = (1+\nu)E[S(t) - I(t)], \\ P_I(t) &= (1+\rho)E[R(t)] - P_R(t) - (\rho - \nu)\mu_X \lambda t + (1+\nu)E[I(t)], \end{aligned}$$

where, $\nu \geq \rho > 0$ is assumed to ensure that the premium charged by the insurer and reinsurer satisfies the no rip-off condition.

3.1 Excess of loss reinsurance

Excess of loss (EOL) reinsurance provides coverage per risk for the large claims over the chosen limit $m < x_F$. EOL cuts down the exposure for the insurer, therefore, it is frequently used in casualty policies. The loss variables, $R_m(t)$ for the reinsurer and $I_m(t)$ for the insurer are defined as $R_m(t) = \sum_{i=1}^{N(t)}(X_i - m)_+$ and $I_m(t) = \sum_{i=1}^{N(t)}(X_i \wedge m)$, where

$$\begin{aligned} (X_i - m)_+ &= \begin{cases} 0, & \text{if } X_i \leq m, \\ X_i - m, & \text{if } X_i > m, \end{cases} \\ X_i \wedge m &= \begin{cases} X_i, & \text{if } X_i \leq m, \\ m, & \text{if } X_i > m, \end{cases} \end{aligned}$$

respectively. Hence, the excess loss above the limit amount, $\{X - m|X > m\}$, will be transferred to the reinsurer. The corresponding distribution function of R_m and I_m are

$$\begin{aligned} F_{R_m}(x) &= F_X(x+m), \qquad x \geq 0, \\ F_{I_m}(x) &= \begin{cases} F_X(x), & \text{if } x \leq m, \\ 1, & \text{if } x > m. \end{cases} \end{aligned}$$

Since, losses above a prescribed threshold u follows a GPD, losses above any $m \geq u$ also follows a GPD. The corresponding expected value premiums, $P_0(t)$, together with their estimators, $\hat{P}_0(t)$, for reinsurer and insurer become Albrecher et al. (2017).

$$\begin{aligned} P_{R_m}(t) &= (1+\nu)E[R(t)] = (1+\nu)\lambda t E[(X_i - m)_+], \\ &= (1+\nu)\lambda t \bar{F}_X(m)e(m), \\ \hat{P}_{R_m}(t) &= (1+\nu)\lambda t \frac{N_u}{n}\big[1+\hat{\kappa}\frac{(m-u)}{\hat{\sigma}}\big]^{-\frac{1}{2}}\big(\frac{\hat{\sigma}}{1-\hat{\kappa}} + \frac{\hat{\kappa}m}{1-\hat{\kappa}}\big), \\ P_{I_m}(t) &= (1+\rho)E[S(t)] - (1+\nu)E\big(P_{R_m}(t)\big) \\ &= (1+\rho)\hat{\mu}_X\lambda t - (1+\nu)\lambda t\bar{F}_X(m)e(m), \\ \hat{P}_{I_m}(t) &= (1+\rho)\hat{\mu}_X\lambda t - (1+\nu)\lambda t\frac{N_u}{n}\big[1+\hat{\kappa}\frac{(m-u)}{\hat{\sigma}}\big]^{-\frac{1}{2}}\big(\frac{\hat{\sigma}}{1-\hat{\kappa}} + \frac{\hat{\kappa}m}{1-\hat{\kappa}}\big), \end{aligned}$$

where $e(m)$, $m \geq u$ is the mean excess function and $\bar{F}_X(m)$ is approximated as given in (9). Note that from (6), $e\big(VaR_q(X)\big) = ES_q(X) - VaR_q(X)$ and if one takes $VaR_q(X) = m$, this leads to

$$\begin{aligned} E\big[X - m)_+\big] &= E\big[\big(X - VaR_q(X)\big)_+\big] \\ &= \bar{F}_X\big(VaR_q(X)\big)E\big[X - VaR_q(X)|X > VaR_q(X)\big] \\ &= \bar{F}_X\big(VaR_q(X)\big)e\big(VaR_q(X)\big) \\ &= \bar{F}_X\big(VaR_q(X)\big)\big(ES_q(X) - VaR_q(X)\big), \end{aligned}$$

therefore, $P_{R_m}(t)$ and $P_{I_m}(t)$ can also be represented by their corresponding VaR and ES measures.

3.2 Quota share reinsurance

Quota share (QUO) is an administratively simple reinsurance contract which shares the loss between the insurance and reinsurance based on a weight parameter, $0 < \beta < 1$. The loss variables R_β of the reinsurer and I_β of the insurer are $R_\beta(t) = (1-\beta)\sum_{i=1}^{N(t)} X_i$ and $I_\beta(t) = \beta\sum_{i=1}^{N(t)} X_i$.

The corresponding distribution functions of R_β and I_β are $F_{R_\beta}(x) = F_X\big(\frac{x}{1-\beta}\big)$ and $F_{I_\beta}(x) = F_X\big(\frac{x}{\beta}\big)$.

The expected value premiums for reinsurer and insurer together with their estimators become

$$P_{R_\beta}(t) = (1+\nu)E[R_\beta(t)] = (1+\nu)\lambda t(1-\beta)\mu_X,$$

$$\begin{aligned}
\hat{P}_{R_\beta}(t) &= (1+\nu)\lambda t(1-\beta)\hat{\mu}_X, \\
P_{I_\beta}(t) &= (1+\rho)E[S(t)] - P_{R_\beta}(t) = \mu_X \lambda t[(\rho-\nu)+\beta(1+\nu)], \\
\hat{P}_{I_\beta}(t) &= \hat{\mu}_X \lambda t[(\rho-\nu)+\beta(1+\nu)], \qquad (10)
\end{aligned}$$

respectively.

3.3 Largest claims reinsurance

Largest claims (LCR) reinsurance transfers the largest r number of claims to the reinsurer that occurred up to time horizon t (see Ammeter (1964)). Therefore, the number of reinsured claims is predefined as a positive integer r. As each new claim arrives, the r largest claims in the most recent moving window of length w such that $\{1 \leq r \leq w\}$, is determined. For any following claim exceeding the previous largest claim, the exceedance amount is paid by the reinsurer.

Let $X^*_{1,N(t)} \geq X^*_{2,N(t)} \geq \cdots \geq X^*_{N(t),N(t)}$ denote the order statistics of X_i. The amount paid by the reinsurer and insurer until time t are defined as

$$\begin{aligned}
R_{lcr}(t) &= \sum_{i=1}^{r} X^*_{i,N(t)} I\big(N(t) \geq r\big), \\
I_{lcr}(t) &= S(t) - \sum_{i=1}^{r} X^*_{i,N(t)} I\big(N(t) \geq r\big),
\end{aligned}$$

respectively. If the largest r claims in a moving window of length w are covered by the LCR, then by the definition of value-at-risk in (5), claims above the level $VaR_{(1-\frac{r}{w})}$ are transferred to the reinsurer. Corresponding distribution functions of R_{lcr} and I_{lcr} are

$$\begin{aligned}
F_{lcr}(x) &= \begin{cases} F_X(VaR_{(1-\frac{r}{w})}), & \text{if } x < VaR_{(1-\frac{r}{w})}, \\ F_X(x), & \text{if } x \geq VaR_{(1-\frac{r}{w})}. \end{cases} \\
F_{lcr}(x) &= \begin{cases} F_X(x), & \text{if } x < VaR_{(1-\frac{r}{w})}, \\ 1, & \text{if } x \geq VaR_{(1-\frac{r}{w})}. \end{cases}
\end{aligned}$$

Therefore, the expected value premiums for reinsurer and insurer together with their estimators can be shown as

$$\begin{aligned}
P_{R_{clr}}(t) &= (1+\nu)\sum_{n=r}^{w} rES_{(1-\frac{r}{n})}(X)\frac{e^{-\lambda t}(\lambda t)^n}{n!}, \\
\hat{P}_{R_{clr}}(t) &= (1+\nu)\sum_{n=r}^{w} r\hat{E}S_{(1-\frac{r}{n})}(X)\frac{e^{-\lambda t}(\lambda t)^n}{n!}, \\
P_{I_{clr}}(t) &= (1+\rho)\mu_X\lambda t - P_{R_{clr}}(t), \\
\hat{P}_{I_{clr}}(t) &= (1+\rho)\hat{\mu}_X\lambda t - (1+\rho)\mu_X\lambda t - P_{R_{clr}}(t), \qquad (11)
\end{aligned}$$

respectively.

3.4 Ecomor reinsurance

Introduced by (Thepaut, 1950), ecomor (ECO) is an excess-of-loss reinsurance where the threshold is determined as the (r^*+1)th largest claim. The number of reinsured claims is predetermined as r^*. If $N(t) \leq r^*$, the random retention level is nonexistent and assumed to be zero, and the reinsurer pays all the claims. Let $X^*_{1,N(t)} \geq X^*_{2,N(t)} \geq \cdots \geq X^*_{N(t),N(t)}$ denote the order statistics of X_i. The amount paid by the reinsurer and insurer are defined as

$$\begin{aligned} R_{eco}(t) &= \sum_{i=1}^{N(t)} \big(X_i - X^*_{r^*+1,N(t)}\big)_+ I\big(N(t) \geq r^*\big) \\ &= \sum_{i=1}^{r^*} X^*_{i,N(t)} I\big(N(t) \geq r^*\big) - r^* X^*_{r^*+1,N(t)} \\ &\quad - r^* X^*_{r^*+1,N(t)} I\big(N(t) \geq r^*\big), \\ I_{eco}(t) &= S(t) - \sum_{i=1}^{N(t)} \big(X_i - X^*_{r^*+1,N(t)}\big)_+ I\big(N(t) \geq r^*\big), \end{aligned}$$

respectively. If the excess of r^* largest claims over the (r^*+1)th are covered by reinsurance, then in the selected moving window of length w, any excess claim amount above $VaR_{1-\frac{r^*+1}{w}}$ is transferred to the reinsurer. Therefore similar to LCR reinsurance, the distribution function for insurance will be a GPD truncated above the $VaR_{1-\frac{r^*+1}{w}}$. In this approach, the distribution functions of R_{eco} and I_{eco} are

$$\begin{aligned} F_{R_{eco}} &= \begin{cases} F_X(VaR_{1-\frac{r^*+1}{w}}), & \text{if } x = 0, \\ F_X(x + VaR_{1-\frac{r^*+1}{w}}), & \text{if } x \geq 0, \end{cases} \\ F_{I_{eco}} &= \begin{cases} F_X(x), & \text{if } x < VaR_{1-\frac{r^*+1}{w}}, \\ 1, & \text{if } x \geq VaR_{1-\frac{r^*+1}{w}}. \end{cases} \end{aligned}$$

respectively, whose expected premiums for reinsurer and insurer together with their estimators are expressed as

$$\begin{aligned} P_{R_{eco}}(t) &= (1+\nu) \sum_{n=r^*}^{w} \big[rES_{(1-\frac{r^*}{n})}(X) - rVaR_{1-\frac{r^*+1}{w}}\big] \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad (12) \\ \hat{P}_{R_{eco}}(t) &= (1+\nu) \sum_{n=r^*}^{w} \big[r\hat{ES}_{(1-\frac{r^*}{n})}(X) - rV\hat{a}R_{1-\frac{r^*+1}{w}}\big] \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \\ P_{I_{eco}}(t) &= (1+\rho)\mu_X \lambda t - P_{R_{eco}}(t), \\ \hat{P}_{I_{eco}}(t) &= (1+\rho)\hat{\mu}_X \lambda t - (1+\nu) \sum_{n=r^*}^{w} \big[r\hat{ES}_{(1-\frac{r^*}{n})}(X) - rV\hat{a}R_{1-\frac{r^*+1}{w}}\big] \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \end{aligned}$$

respectively.

4 Ruin probability

This section demonstrates and compares the possible effects of each aforementioned reinsurance strategy from the perspectives of the reinsurer and the insurer. The main aim of reinsuring any loss branch is to estimate and control the ruin probability. In the classical risk theory, ruin probabilities are commonly derived for light-tailed distributions. On the other hand, our approach derives the asymptotic ruin probabilities for a heavy-tailed data set depending on these reinsurance strategies. Therefore, we provide realistic and accurate risk estimates for the contracts encountered in practice.

Assume that the claim frequency and severity are independent. The corresponding Cramer-Lundberg model Lundberg (1903) for the surplus process is given as

$$U(t) = u_0 + c(t) - S(t), \qquad t \geq 0,$$

where $u_0 \geq 0$, denotes the initial wealth, $c(t) = (1+\rho)E\big[N(t)\big]E(X)$ is the continuous premium income up to time t with loading factor ρ. e assume that the premium satisfies the net profit condition $c(t) > \lambda t \mu_X$, so that the insurer's probability of ruin despite its high initial surplus is not equal to one. Since a Poisson process is assumed for the compound loss distribution, the interarrival time of losses follows an exponential distribution with $\frac{1}{\lambda}$ and it is constant. The ultimate ruin probability, corresponding to $U(t) < 0$ is defined as

$$\psi(u_0) = P\big(U(t) < 0, \text{for} \quad \text{some} \quad t\big).$$

For the light-tailed distributions, where X has a finite moment generating function, and if there exists an adjustment coefficient, $R > 0$, nd the probability of ruin is bounded such that $\psi(u) \leq e^{-Ru}$. or the ruin probability of the heavy-tailed GPD, which moment generating functions do not exist for high moments, the Cramer-Lundberg probability estimates do not exist either. Nonetheless, strict Pareto and Pareto type distributions belong to the class of subexponential distributions, $F \in \mathcal{S}$, and by using their properties one can asymptotically compute the related ruin probabilities (see Albrecher et al. (2017), Bazyari (2023a) and Bazyari (2023b)).

Recalling some heavy-tailed distribution properties, if F belongs to the subexponential distribution class, which is a subclass of heavy-tailed distributions with the tail decreasing slower than that of any exponential distribution, the following equation holds

$$\lim_{x\to\infty} \frac{\bar{F}^{n^*}(x)}{\bar{F}(x)}, \qquad \forall n \geq 2,$$

where, F^{n^*} is the n-fold convolution. (for more details on the Subexponential distribution, the reader can refer to Bazyari (2022)).

Modeling the extreme losses can be considered as

$$P\big(X_1 + \cdots + X_n > x\big) = \bar{F}^{n^*}(x) \sim P\big(\max(X_i) > x\big), \ \ as \ \ x \to \infty,$$

and hence, the asymptotic ruin probability is computed such as

$$\psi(u_0) = \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} (1+\rho)^{-n} \bar{F}_L^{n*}(u_0), \qquad u_0 \geq 0,$$

where, the n-fold convolution of the integrated tail distribution of F or sometimes called as the Lorenz curve, $F_L(x)$, is defined as

$$F_L(x) = \frac{1}{\mu_X}\int_0^x \bar{F}(y)dy, \qquad x \geq 0.$$

Given that, $F \in \mathcal{S}$, one can write for a sufficiently large u_0

$$\frac{\psi(u_0)}{\bar{F}_L(u_0)} = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}(1+\rho)^{-n}\frac{\bar{F}_L^{n*}(u_0)}{\bar{F}_L(u_0)} = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}(1+\rho)^{-n}n = \rho^{-1}.$$

Then, the ruin probability in the renewal risk model becomes

$$F_L \in \mathcal{S} \Longleftrightarrow 1-\psi(u_0) \in \mathcal{S} \Longleftrightarrow \psi(u_0) = \rho^{-1}\bar{F}_L(u_0), \;\; u_0 \to \infty,$$

and can be expressed as

$$\psi(u_0) = \frac{1}{\rho\mu_X}\int_{u_0}^{\infty}\bar{F}(y)dy,$$

which indicates that, even if the initial surplus u_0 is large, ruin may occur depending only on the integrated tail distribution function of F. Also, the ruin probability is not affected by the parameter λ since both the premium and the aggregate loss are affected by it at the same rate. Let us assume that, at time t, the surplus processes for reinsurer, $U_{R(.)}(t)$, and insurer, $U_{I(.)}(t)$, for each reinsurance scheme are defined as

$$\begin{aligned} U_{R(.)}(t) &= u_0 + P_{R(.)}(t) - R(.)(t), \qquad t \geq 0, \\ U_{I(.)}(t) &= u_0 + P_{I(.)}(t) - I(.)(t), \qquad t \geq 0, \end{aligned}$$

where (.) takes $\{m, \beta, lcr, eco\}$ with respect to EOL, QUO, LCR, and ECO, respectively. Similarly, the ruin probability are defined as $\psi_{R(.)}(u_0)$ and $\psi_{I(.)}(u_0)$, respectively. Based on our proposed approach, we develop and analytically derive the surplus processes and ruin probabilities for each reinsurance strategies accordingly.

4.1 Parameter selection

To maintain the comparability between the reinsurance treaties, we select QUO, LCR, and ECO reinsurance parameters, $\{\beta, r, r^*\}$, depending on the EOL reinsurance parameter $m \geq u$ and according to the criteria that the total premium $(P_R + P_I)$ is equal in each treaty. This also provides that the distribution functions of EOL, QUO, LCR and ECO will belong to the subexponential family. For the estimation of the QUO reinsurance parameter β, expected premiums in (10) and (10) are taken as equal which leads to

$$\beta = 1 - \frac{N_u}{\mu_X n}\left[1+\kappa\frac{(m-u)}{\sigma}\right]^{-\frac{1}{\kappa}}\left(\frac{\sigma}{1-\kappa}+\frac{\kappa}{1-\kappa}m\right).$$

(10) and (11) are used in determining the LCR reinsurance parameter r, and (10) and (12) are used in determining the ECO parameter r^*. The minimum integer values of $\{r, r^*\}$ are chosen to satisfy the equations. In order to maintain the comparability of strategies in each time point, the parameter estimations need to be redone in each moving window.

4.2 Under EOL assumption

The surplus process estimations for the reinsurer and the insurer are derived as

$$\begin{aligned}\hat{U}_{R_{(m)}}(t) &= u_0 + (1+\nu)\lambda t \frac{N_u}{n}\big[1+\hat{\kappa}\frac{(m-u)}{\hat{\sigma}}\big]^{-\frac{1}{\hat{\kappa}}}\big(\frac{\hat{\sigma}\hat{\kappa}}{1-\hat{\kappa}}\big) - \sum_{i=1}^{w}(X_i - m)_+,\\ \hat{U}_{I_{(m)}}(t) &= u_0 + (1+\rho)\hat{\mu_X}\lambda t - (1+\nu)\lambda t \frac{N_u}{n}\big[1+\hat{\kappa}\frac{(m-u)}{\hat{\sigma}}\big]^{-\frac{1}{\hat{\kappa}}}\big(\frac{\hat{\sigma}\hat{\kappa}}{1-\hat{\kappa}}\big)\\ &\quad -\sum_{i=1}^{w}(X_i \wedge m),\end{aligned}$$

whose ruin probabilities, for a sufficiently large u_0, become

$$\begin{aligned}\hat{\psi}_{R_{(m)}}(u_0) &= \big[\nu\frac{N_u}{n}\big[1+\hat{\kappa}\frac{(m-u)}{\hat{\sigma}}\big]^{-\frac{1}{\hat{\kappa}}}\big(\frac{\hat{\sigma}}{1-\hat{\kappa}}+\frac{\hat{\kappa}}{1-\hat{\kappa}}m\big)\big]^{-1}\int_{u_0}^{\infty}\bar{F}_{R_m}(y)dy,\\ \hat{\psi}_{I_{(m)}}(u_0) &= \big[\rho\hat{\mu}_X - \nu\frac{N_u}{n}\big[1+\hat{\kappa}\frac{(m-u)}{\hat{\sigma}}\big]^{-\frac{1}{\hat{\kappa}}}\big(\frac{\hat{\sigma}}{1-\hat{\kappa}}+\frac{\hat{\kappa}}{1-\hat{\kappa}}m\big)\big]^{-1}\int_{u_0}^{\infty}\bar{F}_{I_m}(y)dy,\end{aligned}$$

respectively.

4.3 Under QUO assumption

The estimates for surplus process regarding to the reinsurer and the insurer are

$$\begin{aligned}\hat{U}_{R_\beta}(t) &= u_0 + (1+\nu)(1-\beta)\hat{\mu}_X\lambda t - (1-\beta)\sum_{i=1}^{w}X_i,\\ \hat{U}_{I_\beta}(t) &= u_0 + \big((\rho-\nu)+\beta(1+\nu)\big)\hat{\mu}_X\lambda t - \beta\sum_{i=1}^{w}X_i,\end{aligned}$$

whose ruin probabilities, for a sufficiently large u_0, become

$$\begin{aligned}\hat{\psi}_{R_\beta}(t) &= \big[\nu(1-\beta)\hat{\mu}_X\big]^{-1}\int_{u_0}^{\infty}\bar{F}_{R_\beta}(y)dy,\\ \hat{\psi}_{I_\beta}(t) &= \big[\big((\rho-\nu)+\beta\nu\big)\hat{\mu}_X\big]^{-1}\int_{u_0}^{\infty}\bar{F}_{I_\beta}(y)dy,\end{aligned}$$

respectively.

4.4 Under LCR assumption

The surplus process for the reinsurer and the insurer are estimated as

$$\begin{aligned}\hat{U}_{R_{lcr}}(t) &= u_0 + (1+\nu)\sum_{n=r}^{w} r\hat{E}S_{(1-\frac{r^*}{n})}(X)\frac{e^{-\lambda t}(\lambda t)^n}{n!} - \sum_{i=1}^{r}X^*_{i,N(t)},\\ \hat{U}_{I_{lcr}}(t) &= u_0 + (1+\rho)\hat{\mu}_X\lambda t - (1+\nu)\sum_{n=r^*}^{w} r\hat{E}S_{(1-\frac{r^*}{n})}(X)\frac{e^{-\lambda t}(\lambda t)^n}{n!}\end{aligned}$$

$$-S(t) + \sum_{i=1}^{r} X^*_{i,N(t)},$$

Accordingly, the ruin probabilities for a sufficiently large u_0 become

$$\begin{aligned}
\hat{U}_{R_{lcr}}(t) &= \big(\nu \sum_{n=r}^{w} r\hat{E}S_{(1-\frac{r^*}{n})}(X) \frac{e^{-\lambda t}(\lambda t)^n}{n!}\big)^{-1} \int_{u_0}^{\infty} \bar{F}_{R_{lcr}}(y)dy, \\
\hat{U}_{I_{lcr}}(t) &= \big(\rho\hat{\mu}_X \lambda t - \nu \sum_{n=r^*}^{w} r\hat{E}S_{(1-\frac{r^*}{n})}(X) \frac{e^{-\lambda t}(\lambda t)^n}{n!}\big)^{-1} \int_{u_0}^{\infty} \bar{F}_{I_{lcr}}(y)dy,
\end{aligned}$$

respectively.

4.5 Under ECO assumption

The surplus process estimations for the reinsurer and the insurer are

$$\begin{aligned}
\hat{U}_{R_{eco}}(t) &= u_0 + (1+\nu) \sum_{n=r^*}^{w} \big[r^*\hat{E}S_{(1-\frac{r^*}{n})}(X) - r^* Va\hat{R}_{1-\frac{r^*+1}{n}}\big] \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&\quad - \sum_{i=1}^{r} X^*_{i,N(t)} - r^* X^*_{r^*+1,N(t)}, \\
\hat{U}_{I_{lcr}}(t) &= u_0 + (1+\rho)\hat{\mu}_X \lambda t - (1+\nu) \sum_{n=r^*}^{w} \big[r\hat{E}S_{(1-\frac{r^*}{n})}(X) \\
&\quad - r^* Va\hat{R}_{1-\frac{r^*+1}{n}}\big] \frac{e^{-\lambda t}(\lambda t)^n}{n!} - \sum_{i=1}^{w} \big(X_i - \big(X_i - X^*_{r^*+1,N(t)}\big)_+\big).
\end{aligned}$$

And the ruin probabilities, for a sufficiently large u_0, become

$$\begin{aligned}
\hat{U}_{R_{eco}}(t) &= \big(\nu \sum_{n=r^*}^{w} r^*\hat{E}S_{(1-\frac{r^*}{n})}(X) - r^* V\hat{a}R_{\left(1-\frac{r^*+1}{n}\right)} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\big)^{-1} \int_{u_0}^{\infty} \bar{F}_{R_{eco}}(y)dy, \\
\hat{U}_{I_{eco}}(t) &= \big(\rho\hat{\mu}_X \lambda t - \nu \sum_{n=r^*}^{w} r^*\hat{E}S_{(1-\frac{r^*}{n})}(X) \\
&\quad - r V\hat{a}R_{\left(1-\frac{r^*+1}{n}\right)} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\big)^{-1} \int_{u_0}^{\infty} \bar{F}_{I_{eco}}(y)dy,
\end{aligned}$$

respectively.

5 Application

We consider an open data source on non-life insurance provided by the US Insurance Services Office whose details are kept anonymous Frees and Valdez (1998). The data set contains 1,500 general liability claims (Loss) and their corresponding allocated loss adjustment expenses (ALAE), which provides sound reasoning for the applicability of all four reinsurance strate-gies. ALAE covers mainly the fees paid to outside attorneys, medical consultants, insurance experts, and legal fees. This data set is studied extensively in copula literature and is reckoned to create a presentable base for dependence illustration (for more details we refer to Kulekci et al. (2023) and Denuit et al. (2004)). We removed 34 claims from the data set, which were left truncated and censored due to the policy limits and deductibles. In Figure 1, the remaining $n = 1,466$ data from both series show recognizable peak values at certain times.

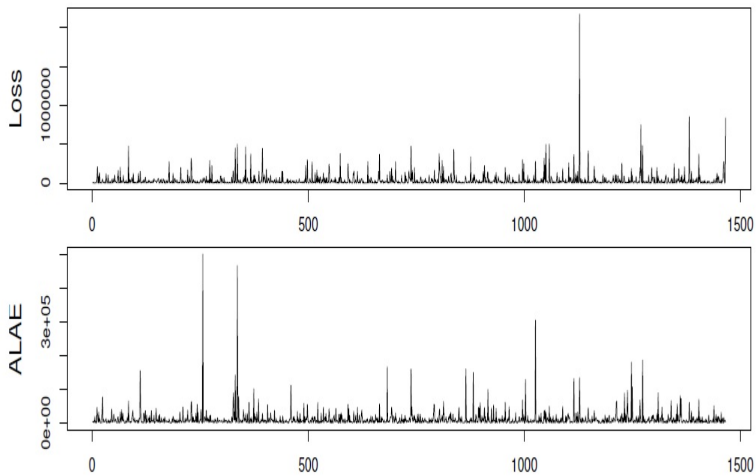


Figure 1: Loss and ALAE in the original scale.

Descriptive statistics in Table 1 depict that the observations are right skewed and have high peakedness.

Table 1: Descriptive statistics of Loss and ALAE.

Variable	Min.	Q_2	Median	Mean	Q_3	Max.	Skew.	Kurt.	Std. Dev.
Loss	10	3750	11048	37110	32000	2173595	10.97	209.91	9251280
ALAE	15	2318	5420	12018	12292	501863	10.8	152.60	2671235

It is experienced that if the dynamic window length (w) is selected as 100, there remains an insufficient number of observations above the threshold for parameter estimation, and the GPD assumption weakens. If w is increased to 500, the real extreme values in the data are not reflected enough in the risk measure estimations. For these reasons, we take the moving window length of 250 observations in estimating the model parameters. The Akaike information criterion (AIC), and the Bayesian information criterion (BIC) show that the best-fitted time series model, amongst $ARMA(p, q) - GARCH(r, s)$ with $\{p, q, r, s \leq 2\}$, is indeed $ARMA(1, 1) - GARCH(1, 1)$. The estimated shape parameter of GPD becomes approximately linear around 80% of the ordered data. Therefore, the POT threshold u in each moving window is assigned corresponding to this quantile. One-step-ahead risk measure estimations are computed for the commonly used $q = 0.95$ onfidence level with estimated parameters using 10^4

simulations.

We assume the policy limit m for the EOL reinsurance in each moving window as the 80% quantile. The remaining reinsurance parameters $\{\beta, r, r^*\}$ f QUO, LCR, and ECO are estimated accordingly, with the constraint that the total premium for the insured in each reinsurance treaty remains equal. We assume loading factors $\{\rho, \nu\} = \{0.15, 0.2\}$ or the insurer and reinsurer, respectively, which are chosen arbitrarily with the condition $\nu\ ge \rho$ to ensure the no rip-off condition. To observe the impact of $\{\rho, \nu\}$, on results, different values are also considered, and it is found to be insignificant in the choice of optimal reinsurance strategies. Throughout the application part, the dashed lines in the figures represent the reinsurer, and solid lines define the insurer. The λ alue is assumed as 1. Choosing a different λ only changes the surplus value at the same rate and does not affect the order of optimal reinsurance based on the ruin probability order.

The dependence structure is explained in two parts. We illustrate the performance of the dynamic GARCH-EVT model with and without the assumption of copula to evaluate the effect of dependence. Since extreme events' impact on a portfolio may seem less prominent, a portfolio consisting of equally weighted Loss-ALAE was also tested alongside the individual Loss and ALAE data sets. The simplified implementation steps are presented in Table 2 to follow the combined modeling stages.

Figure 2 shows the dynamic GPD parameter estimations via pseudo-MLE for the GARCH-EVT and GARCH-EVT-copula models. The estimated κ parameters are largely below zero, indicating that the limiting distribution for normalized maximas, generalized under GPD, arises from a Weibull extreme value distribution. Transitioning from GARCH-EVT to GARCH-EVT-copula, the effects of incorporated dependence show up as a decrease in $\hat{\kappa}$ and $\hat{\sigma}$. The change is more prominent for the heavier-tailed ALAE.

5.1 GARCH-EVT model

This section employs the GARCH-EVT framework without the copula assumption. (7) and (8) are utilized to calculate $VaR_{0.95}$ and $ES_{0.95}$ estimates, as presented in Figure 3 for Loss, ALAE, and the equally-weighted Portfolio, respectively. The jump responses of risk measures are visible following significantly high losses to cover the potential future extremes. A violation is defined as a realized claim exceeding the estimated one-step ahead risk measure. Employing a moving window of 250 data results in 1216 estimation points, and with a 0.95 confidence level, the expected number of violations for the risk measures is 60.8.

To assure the reliability of estimated risk measures, we employ three backtesting methods. The individual and portfolio backtesting results for actual violations are presented in Table 3. Based on the unconditional coverage (UC) and conditional coverage (CC) tests, the number of $VaR_{0.95}$ violations for Loss, ALAE, and Portfolio falls within the accepted limits, and they are independent (Kupiec, 1995 and Christoffersen,1998). ALAE containing more extremes than the others, shows a relatively poorer fit with 49 violations, compared to Loss and the Portfolio. The results of the residuals bootstrap test for $ES_{0.95}$ state that, for both individual and portfolio cases, the mean of excess violations of $VaR_{0.95}$ is greater than zero (McNeil and Frey, 2000). The backtesting results support the justifiability of the GARCH-EVT model.

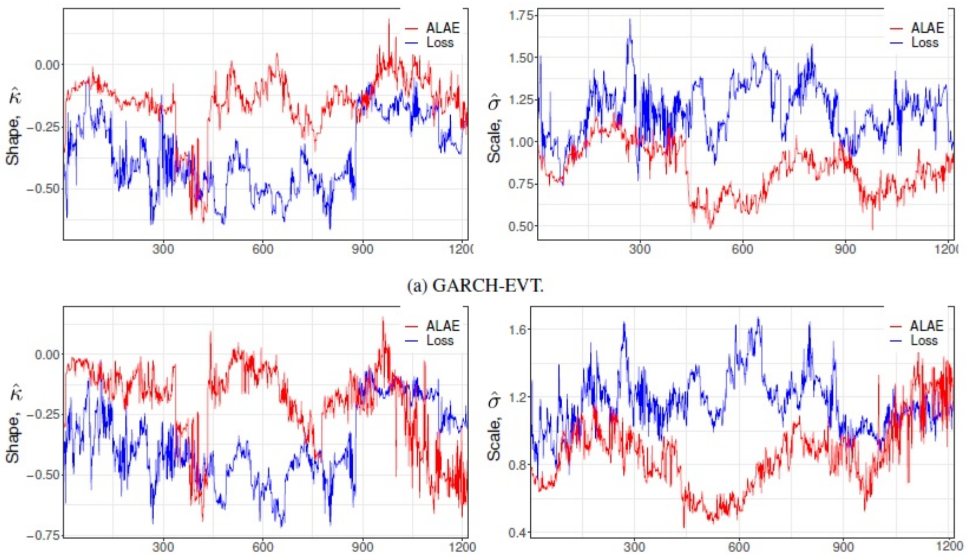


Figure 2: GPD parameter estimations.

Table 2: Algorithm of the dynamic EVT-GARCH-copula model.

Input: Log-return data set, $\{n, w, q\} = \{1466, 250, 0.95\}$

for $j = \{Loss, ALAE, Portfolio\}$ do
for $k = 1 : (n - w + 1)$ do
for $i = k : (k + w + 1)$ do
Estimate ARMA(1, 1)-GARCH(1, 1) parameters.
Obtain standardized residuals (std. res.).
Convert std. res. to uniform std. res.
If integrate copula then
Fit copula to uniform std. res. and simulate 10^4 one-step-ahead estimates.
Back transform simulated uniform std. res. to simulated std. res. using the inverse cdf.
end
Apply EVT to simulated std. res.
Select threshold value, and estimate GPD parameters.
end
Estimate residual VaR_ϵ and ES_ϵ using GPD parameters.
Estimate VaR_q and ES_q using ARMA(1, 1)-GARCH(1, 1) coefficients.
end
Compute premium estimations, $\hat{P}_R$ and $\hat{P}_I$, for j.
Compute premium estimations, $\hat{U}_R$ and $\hat{U}_I$, for j.
Compute the heavy tailed ruin probabilities, $\hat{\psi}_R(u_0)$ and $\hat{\psi}_I(u_0)$ for j.
end

The dynamically estimated premium values under the four reinsurance strategies are presented in Figure 4 for both the insurer and the reinsurer. While EOL and QUO exhibit relatively stable premium values that stay close to each other, LCR and ECO show more variability in premium values, especially after the occurrence of an extreme

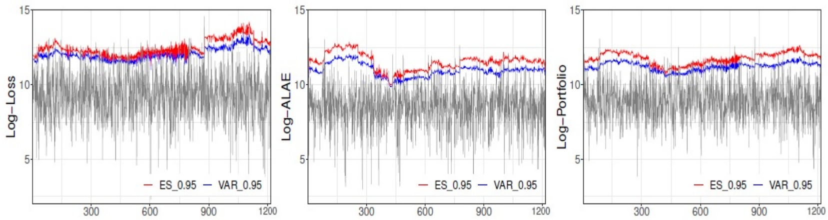


Figure 3: Risk measure estimations in the GARCH-EVT setting.

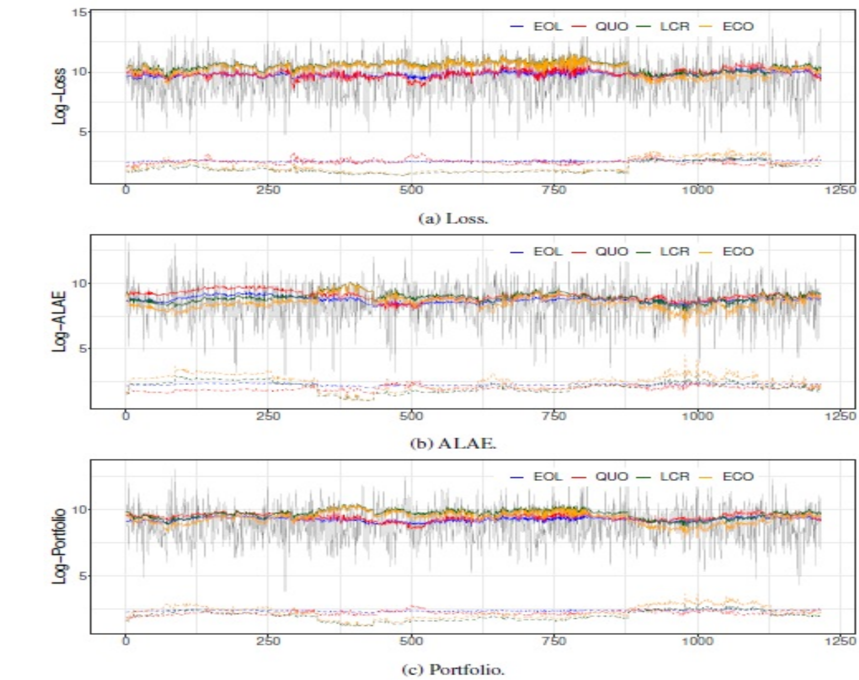
Table 3: Backtesting results in the GARCH-EVT setting.

		Loss	ALAE	Portfolio
$VaR_{0.95}$	Actual of violations	60	49	57
	UC p-value	0.9160	0.1086	0.6135
	CC p-value	0.9931	0.1137	0.8619
$ES_{0.95}$	p-value	0	0	0

loss. Under EOL, LCR, and ECO strategies, the reinsurer covers the low-probability and high-impact claims within the data series. In the case of QUO reinsurance, the reinsurer covers a small portion of each claim. Therefore, the premium for the reinsurer is significantly lower than the insurers in each strategy. Since the total premium remains constant in each strategy throughout the testing period, the insurer premium values have the opposite order of what is received for the reinsurers. However, since the level of risk does not remain constant, the selection of the optimal reinsurance strategy should not be solely based on the premium amounts received by the parties.

The realized surplus processes for Loss, ALAE, and the Portfolio are given in Figure 5 for $u_0 = 0$, where the solid lines represent the insurer, while the dashed lines represent the reinsurer. Increasing u_0, the initial surplus value, results in higher surplus levels but does not alter the order of magnitude of the surplus values or the ruin probabilities associated with the reinsurance strategies. For both univariate and portfolio cases, the insurance contracts are ranked from highest to lowest surplus for the insurer as EOL, QUO, LCR, and ECO. The ranking for the reinsurer is the opposite of the insurer's, as expected. Regarding ALAE, LCR and ECO are significantly more advantageous for the reinsurer than EOL and QUO. This is because LCR and ECO provide greater protection against the most extreme values in the process. Contrary to the changing order of the premiums in the univariate and portfolio cases, the same order of surplus values is preserved across all reinsurance treaties. This shows that the premium amounts received by the reinsurer and insurer may not necessarily indicate the preference of a reinsurance contract. The outcome depends heavily on and is shaped by the realized data. The average ruin probabilities over the entire testing period are given in Table 4 for comparison purposes, ranked from the highest (1) to the lowest (4). For the reinsurer, based on the average lowest ruin probability, the optimal treaties are QUO for Loss, which provides a safer position by shared coverage, and ECO for ALAE and Portfolio, which offers greater protection in case of heavier claims. For the insurer, the optimal treaty is EOL for Loss and Portfolio, which can provide a more predictable claim payment process by transferring all excesses over the threshold. For ALAE, the optimal treaty is QUO, which can only lower the ruin probability by sharing all claims.

Optimality is not solely related to the premium received from each strategy, based on the comparison between the premiums and ruin probabilities. For example, EOL results in the highest ruin probabilities for the reinsurer in each case, therefore the highest risk. However, it does not consistently deliver the highest premium.



(a) Loss.

(b) ALAE.

(c) Portfolio.

Figure 4: Premium estimations in the GARCH-EVT setting.

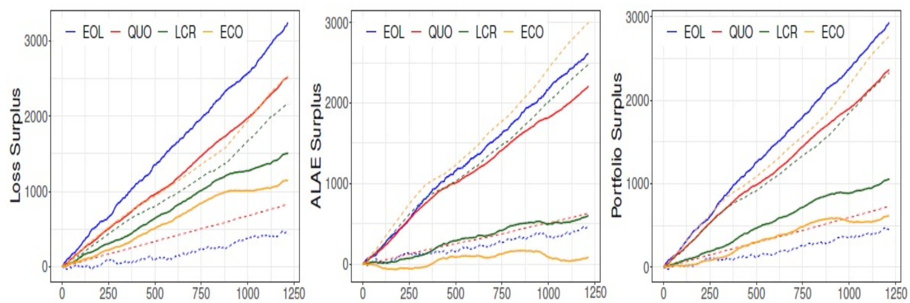


Figure 5: Surplus estimations in the GARCH-EVT setting.

Table 4: Average ruin probabilities in the GARCH-EVT setting.

	Loss		ALAE		Portfolio	
Contract	Reinsurer	Insurer	Reinsurer	Insurer	Reinsurer	Insurer
EOL	0.2234 (1)	0.1726 (4)	0.2483 (1)	0.1688 (3)	0.2359 (1)	0.1707 (4)
QUO	0.0491 (4)	0.1915 (3)	0.0985 (2)	0.1631 (4)	0.0738 (3)	0.1773 (3)
LCR	0.1625 (2)	0.2573 (1)	0.0916 (3)	0.2110 (1)	0.1270 (2)	0.2342 (1)
ECO	0.0896 (3)	0.2248 (2)	0.0299 (4)	0.1794 (2)	0.0597 (4)	0.2022 (2)

Ranking: (1) highest to (4) lowest.

5.2 GARCH-EVT-copula model

We begin by examining dependence in Figure 6 and Figure 7. The scatterplot (Figure 6a) shows a significant and positive correlation between Loss and ALAE. The correlation coefficients (Figure 6b), calculated using a 250-day mowing win-dow, confirm varying positive dependence, ranging from 0.25 to 0.54. This interdependence is more pronounced in the right tail of the distribution, indicating a higher likelihood of an extreme Loss accompanying an extreme ALAE. This is expected due to higher losses often leading to higher expenses. Given these findings, it becomes important to enhance our modeling framework with copula methodologies.

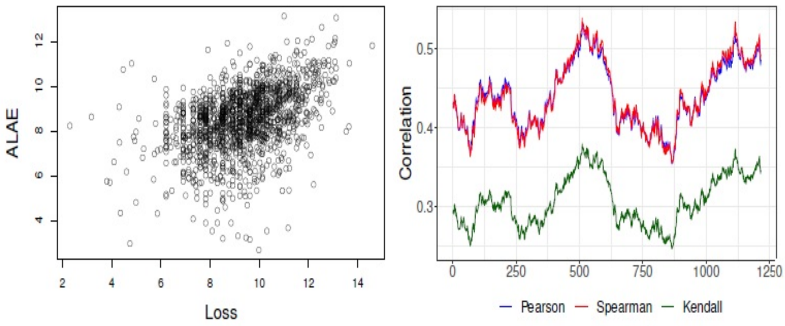
In Figure 8, the tail dependence coefficients indicate that, compared to a Gaussian distribution, left-tail dependence is lower while right-tail dependency is stronger. To determine the optimal copula model among various copula models with different dependence structures, we use maximum likelihood estimation. The results show that the Joe copula, known for its asymmetric and high right-tail dependence from the Archimedean family, is the best-fitting choice. Consequently, we re-estimate ARMA-GARCH, GPD, and Joe copula parameters for each moving window. The $VaR_{0.95}$ and $ES_{0.95}$ estimations of the GARCH-EVT-copula model (in Figure 8) show a slightly more flexible fit compared to those of the GARCH-EVT model (in Figure 2). This improvement is attributed to the incorporation of dependence through the Joe copula. The backtesting results presented in Table 5 support this improvement, especially in terms of the number of violations for ALAE, by resulting in a better fit.

Table 5: Backtesting results in the GARCH-EVT-copula setting.

		Loss	ALAE	Portfolio
	Actual of violations	61	58	63
$VaR_{0.95}$	UC p-value	0.9790	0.7105	0.7735
	CC p-value	0.9996	0.4121	0.8803
$ES_{0.95}$	p-value	0	0	0

It is noticeable that with the addition of copula in the GARCH-EVT-copula model, premiums for Loss and ALAE (in Figure 9a and Figure 9b) increase in EOL and QUO reinsurances, while they decrease in LCR and ECO. The increase in EOL and QUO is indicative of the previously unrecognized extreme right-tail dependence present in the data, which the copula effectively captures. Premium estimations for the Portfolio with the GARCH-EVT-copula model (in Figure 9c) align with the univariate case results. The inclusion of copula increases the average reinsurer premium values for EOL and QUO reinsurances and decreases them for LCR and ECO reinsurances.

The surplus estimations under the GARCH-EVT-copula setup in Figure 10 display



(a) Scatter plot of Loss and ALAE. (b) Correlation coefficients.

Figure 6: Dependence analyses.

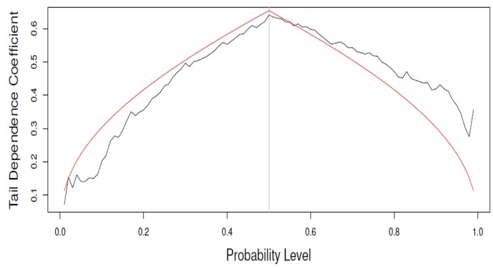


Figure 7: Left and right tail dependencies compared to Gaussian distributions.

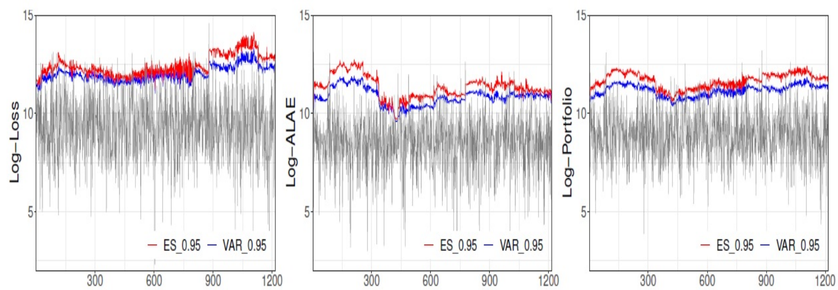


Figure 8: Risk measure estimations in the GARCH-EVT-copula setting.

a decrease in each reinsurance strategy for ALAE and Portfolio when compared to the GARCH-EVT model. This decrease is visibly distinct in the reinsurer surpluses of ALAE due to the improvement in the risk measures and premium estimations by the Joe copula. In the Portfolio, the decline in surplus values is largely influenced by the decrease in ALAE. Surplus estimations for Loss remain roughly the same with the GARCH-EVT model. The reinsurer derives greater benefits from the reduced ruin probability compared to the insurer. Because by modeling the dependence, the copula model accounts for the co-dependency in the extreme values.

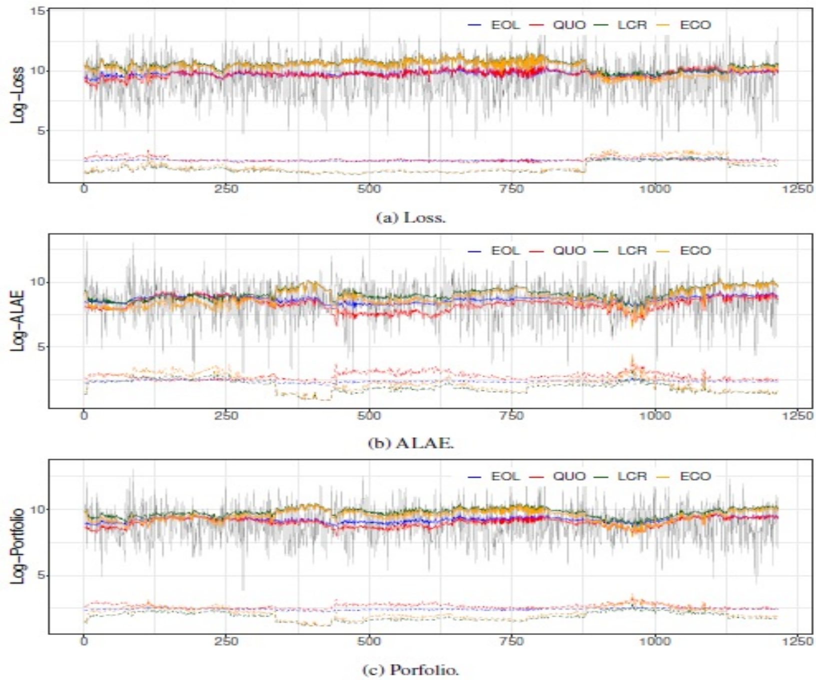


Figure 9: Premium estimations in the GARCH-EVT-copula setting.

In Table 6, it is observed that the average ruin probabilities for Loss maintain a similar order of magnitude as in the GARCH-EVT model. However, for ALAE and Portfolio, the order differs. In ALAE, the optimal treaty for the insurer shifts from QUO to EOL to provide more coverage against the extremes. Conversely, the optimal treaty for the reinsurer remains ECO since ALAE has more extreme data than Loss. As for the Portfolio, the insurer's optimal treaty remains unchanged, while for the reinsurer, it shifts from ECO to QUO. This change arises from Loss returns being higher and more prominent in the Portfolio compared to ALAE.

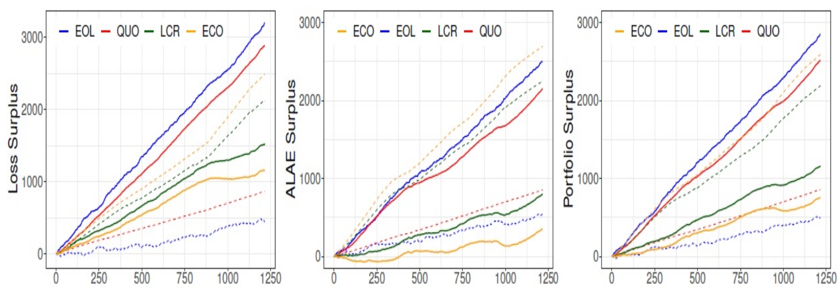


Figure 10: Surplus estimations in the GARCH-EVT-copula setting.

Table 6: Average ruin probabilities in the GARCH-EVT-copula setting.

	Loss		ALAE		Portfolio	
Contract	Reinsurer	Insurer	Reinsurer	Insurer	Reinsurer	Insurer
EOL	0.2044 (1)	0.1648 (4)	0.2040 (1)	0.1569 (4)	0.2042 (1)	0.1609 (4)
QUO	0.0399 (4)	0.1959 (3)	0.0535 (3)	0.2225 (1)	0.0466 (4)	0.2092 (2)
LCR	0.1658 (2)	0.2533 (1)	0.1122 (2)	0.2123 (2)	0.1391 (2)	0.2328 (1)
ECO	0.0933 (3)	0.2224 (2)	0.0481 (4)	0.1807 (3)	0.0708 (3)	0.2016 (3)

Ranking: (1) highest to (4) lowest.

Discussion and conclusions

This paper investigates the impact of reinsurance strategies on the ruin probability in the context of dependent and heavy-tailed actuarial data using the extreme value theory framework. We employ the GARCH-EVT-copula model with a moving windows approach to dynamically estimate one-day-ahead losses. The considered reinsurance treaties include excess-of-loss, quota-share, largest claims, and ecomor. To the best of our knowledge, this is the first study to derive surplus and asymptotic ruin probability estimates under the proposed model, taking into account both the insurer's and reinsurer's perspectives. We employ real-life bivariate insurance data, Loss-ALAE, to demonstrate the influence of the copula. We assess the effects of each reinsurance strategy on p remiums, surplus p rocesses, and ruin p robabilities, both for univariate data and an equally-weighted portfolio. Experimental results using the heavy-tailed Loss-ALAE data reveal that the ARMA(1,1)-GARCH(1,1) model under the peaks over threshold (POT) approach with the Joe copula function is the most suitable model. Findings indicate that the dynamic GARCH-EVT-Copula model can yield more accurate risk assessments and help the insurer and reinsurer to control their risks by facilitating the selection of the appropriate reinsurance strategy.

By incorporating tail dependence, the copula mitigates the over- and under- estimation of the risk due to extreme values. Notably, the largest claims and ecomor reinsurance treaties prove advantageous for reinsurer's risk assessment in heavy-tailed loss scenarios. Ignoring dependence in modeling leads to an underestimation of risk in the case of largest claims and ecomor reinsurances, while overestimating risk in excess-of-loss and quota share arrangements. Furthermore, it's worth noting that, the ruin probabilities for largest claims reinsurance consistently dominate those of ecomor in heavy-tailed data. As future work, introducing the reinvestment of surplus with a stochastic interest rate can make the model even more realistic. Expanding the dimensionality using vine copulas and allowing for changes in the type of reinsurance treaty during the policy period can further improve risk assessment capabilities.

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