Journal of Statistical Modelling: Theory and Applications
Vol. 4, No. 1, 2023, pp. 157-172
Yazd University Press 2023

JSMTA

Research Paper

On dependence of the weighted Marshall-Olkin bivariate exponential model in the presence of the copula function

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Received: January 10, 2024/ Revised: April 13, 2024/ Accepted: May 07, 2024

Abstract: In this paper, we develop a version of the weighted Marshall-Olkin bivariate exponential model by incorporating a new parameter. This parameter describes the dependence structure between margins via a copula function. We choose the inference for margins method to estimate the model parameters along with the copula parameter, as this method offers more advantages than the maximum likelihood estimation method. Additionally, we conduct a comprehensive simulation study to investigate the behavior of the copula parameter estimator and the remaining parameters. Finally, an analysis of a real dataset on automobile insurance reveals that the Clayton copula characterizes the dependence structure within the Archimedean copula family.

Keywords: Archimedean copula; Dependency; Inference for margins method; Optimization; Weighted Marshall-Olkin bivariate exponential.
Mathematics Subject Classification (2010): 62F10, 62P05.

1 Introduction

Often, researchers find that the existing families of distributions are insufficient to meet the diverse requirements encountered in their work. The exponential distribution is notable example in this context. Widely used in life-testing experiments due to its simplicity and lack of memoryless properties, Riad et al. (2022), the exponential distribution does have a limitation: it features a constant hazard rate function. Consequently, it cannot be effectively employed to model data with nonconstant hazard rates. This limitation prompted scholars to explore extensions of this distribution. A significant contribution in this direction comes from Gupta and Kundu (2009) who

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proposed a novel version of the exponential distribution by introducing the shape parameter $\alpha > 0$. This asymmetrical distribution is termed the "weighted exponential (WE) distribution". Due to its well-defined cumulative distribution function (CDF), it can be efficiently employed for the analysis of financial and lifetime data. The probability distribution function (PDF) of two-parameter WE distribution with parameters $\alpha > 0$ and $\lambda > 0$ is as follows:

$$f_X(x;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\alpha\lambda x}), \ \ x > 0.$$

Additionally, the CDF of the WE distribution is expressed as:

$$F_X(x;\alpha,\lambda) = 1 + \frac{1}{\alpha}e^{-\lambda(\alpha+1)x} - \frac{\alpha+1}{\alpha}e^{-\lambda x}, \ \ x > 0.$$

During recent years, multivariate and mixed forms with other distributions have been of great interest for WE distribution. Gupta and Kundu (2009) and Shahbaz et al. (2010) suggested the three-parameter weighted Weibull (WW) model which are common generalization of the WE distribution. Mahdavi et al. (2017) introduced a bivariate weighted exponential distribution based on the generalized exponential distribution. Iqbal and Iqbal (2020) investigated a finite mixture model based on weighted versions of exponential and gamma distributions. Additionally, they explored two real life data applications with this model. Tapan (2022) proposed a new discrete weighted exponential distribution using a special discrete method, and applied the new distribution applied to two real data sets.

In reliability and survival analysis, Marshall and Olkin (1997) proposed a mathematical transformation aimed to expand a family of exponential and Weibull distributions. This transformation facilitates the application of standard probability distributions, including exponential, gamma, and Weibull distributions, as well as statistical methods, thereby simplifying complex analyses. The resulting family is termed the Marshall-Olkin (MO) family of distributions. This renders it a valuable tool in fields such as insurance risk modeling, survival analysis and lifetime data, Rubio and Steel (2012).

A new class of WE distribution using the MO trasformation, namely weighted Marshall-Olkin bivariate exponential (WMOBE) is introduced by Jamalizadeh and Kundu (2013). They introduced WMOBE distribution, using a method that was analogous to that of Azzalini (1985). The singular WMOBE distribution has four parameters and can also be achieved as a hidden truncation model analogous to that of Arnold and Beaver (2000). Arnold et al. (2002) carried out the interpretation of any multivariate hidden truncation model, and so it also was reliable for the suggested model (WMOBE). That's why it was a fundamental stimulant of their proposed model. Besides, Kotz et al. (2019) pointed out that the most popular singular bivariate distribution is the three-parameter Marshall-Olkin bivariate exponential (MOBE) distribution or the four-parameter Marshall-Olkin bivariate Weibull (MOBW) distribution. Because it has been found that the WE distribution may prepare a better fit than the Weibull or exponential in some instances (see Gupta and Kundu (2009)), it is expected that the WMOBE model may also provide a better fit than the MOBE or MOBW model in some instances. Of course, this expectation has been satisfied by Jamalizadeh and Kundu (2013). Moreover, Khan and Kumar (2016) obtained the distribution of concomitant order statistics from the WMOBE distribution. Recent work by Al-Mutairi et al. (2018) focused on various properties of the WW distribution proposed by Shahbaz et al. (2010), extending it to the bivariate and multivariate versions, termed the bivariate weighted Weibull (BWW) distribution and the multivariate weighted Weibull (MWW) distribution.

In parametric models, the most natural frequentist's estimator is typically obtained through the maximum likelihood (ML) method. Kazemi et al. (2021) introduced a new form of WE distribution with four parameters and subsequently derived ML estimates for these parameters, demonstrating its practical application. Building on this, Fallah and Kazemi (2022) addressed a natural generalization of the WE distribution, focusing on inferential aspects such as ML estimators and confidence intervals for the parameters. Riad et al. (2022) studied WE distribution, exploring mathematical features and employing nine classical and approximating Bayesian approaches for estimating the model's parameters. However, in high-dimensional parameter spaces, maximum likelihood imposes a high computational burden. The solution lies in using the inference for margins (IFM) method. This approach can significantly reduce computational complexity, thereby making it feasible to analyze large datasets or models with many parameters.

One of the most critical issues concerning multivariate distributions involves modeling the dependence structure, particularly in financial and economic applications. While it is commonly assumed that the marginals in multivariate distributions are independent, this assumption does not always hold true. Copula functions, a fundamental tool in statistics, are employed to describe the dependence between two or more random variables. These are famous for their simplicity and flexibility in capturing dependence, accommodating arbitrary linear and nonlinear relationships between variables. Over the last two decades, there has been a growing trend among researchers to use copula function for modeling the dependence structure in multivariate distributions. This modeling approach involves coupling the joint distribution functions of random variables with their marginals. The main difference between copula and non-copula models lies in how they model the dependence structure between variables: copula models offer more flexibility by separating the marginal distributions from the dependence structure, while non-copula models often make more restrictive assumptions about the joint distribution of variables. Also, the non-copula models can not capture complex dependencies as effectively as copula models.

In this paper, our objective is to investigate the WMOBE distribution while modeling the dependence structure using the archimedean copula. There is a deficiency regarding the use of the copula function for modeling dependency in the WMOBE model. Subsequently, we estimate the model parameters using the IFM method and evaluate the efficiency of this approach through a simulation study. Finally, we apply the proposed model to a real dataset.

The paper's structure is organized as follows. In Section 2, a brief review of the WMOBE model is provided. Section 3 is devoted to the exploration of copula functions. The estimation of copula models is carried out in Section 4. Section 5 assesses the performance of the proposed estimation method through an extensive simulation study. The application of the proposed model to a real dataset is illustrated in Section 6. Finally, the paper concludes in Section 7, summarizing key findings and contributions.

2 A review on WMOBE model

This section defines and characterizes the WMOBE model perfectly. The random vector (X_1, X_2) follows the WMOBE distribution if the joint PDF of (X_1, X_2) is as follows (Jamalizadeh and Kundu, 2013):

$$g(x_1, x_2) = \begin{cases} g_1(x_1, x_2), & \text{if } x_2 > x_1 > 0, \\ g_2(x_1, x_2), & \text{if } x_1 > x_2 > 0, \\ g_0(x), & \text{if } x_1 = x_2 = x > 0. \end{cases} \quad (1)$$

in which

$$\begin{aligned} g_1(x_1, x_2) &= \frac{\alpha + \lambda}{\alpha} \lambda_1 \exp(-\lambda_1 x_1)(\lambda_2 + \lambda_{12}) \exp(-(\lambda_2 + \lambda_{12})x_2)(1 - \exp(-x_1\alpha)), \\ g_2(x_1, x_2) &= \frac{\alpha + \lambda}{\alpha} (\lambda_1 + \lambda_{12}) \exp(-(\lambda_1 + \lambda_{12})x_1)\lambda_2 \exp(-(\lambda_2 x_2)(1 - \exp(-x_2\alpha)), \\ g_0(x) &= \frac{\alpha + \lambda}{\alpha} \lambda_{12} \exp(-\lambda x)(1 - \exp(-x\alpha)), \end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, and $\Theta = (\alpha, \lambda_1, \lambda_2, \lambda_{12})$ is the parameter vector of the model. α is the shape parameter and $\lambda_1, \lambda_2, \lambda_{12}$ are the scale parameters. From now on, a WMOBE distribution with PDF (1) will be denoted by WMOBE$(\alpha, \lambda_1, \lambda_2, \lambda_{12})$.

Theorem 2.1 (Jamalizadeh and Kundu (2013)). *Let* $(X_1, X_2) \sim WMOBE(\alpha, \lambda_1, \lambda_2, \lambda_{12})$, *then*
1. $X_1 \sim WE(a_1, b_1)$, *which* $a_1 = (\alpha + \lambda_2)/(\lambda_1 + \lambda_{12})$, $b_1 = \lambda_1 + \lambda_{12}$
2. $X_2 \sim WE(a_2, b_2)$, *which* $a_2 = (\alpha + \lambda_1)/(\lambda_2 + \lambda_{12})$, $b_2 = \lambda_2 + \lambda_{12}$.

3 Copula functions

Modeling dependence is crucial in multivariate statistics and research fields such as actuarial sciences and finance. Copulas have become increasingly popular for this purpose, Cherubini et al. (2004). Copulas are linking functions that connect multivariate distributions to their one-dimensional margins. Their popularities stems from the flexibility and modeling possibilities they offer for dependence. Copulas were first introduced by Sklar in 1959, but they have gained more prominence in the literature since 1999, Genest et al. (2013). Sklar (1959) decomposes the joint distribution functions of random variables into their marginals and dependence structure modeled by the copula function C:

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

This decomposition is helpful for two reasons:
1- It enables the use of any univariate marginal distributions.
2- It allows the modeling of heavy tails and sophisticated dependencies.

For a bivariate, continuous random variable $\mathbf{X}$ with a distribution function F and univariate continuous margins F_j ; $j = 1, 2$, the copula function C is a distribution function $C : [0, 1]^2 \longrightarrow [0, 1]$ with $U(0, 1)$ margins that satisfies

$$F(X_1, X_2) = C(F_1(X_1), F_2(X_2)).$$

If the univariate margins are absolutely continuous with respective densities $f_j = F_j'$ and if C has mixed derivatives of order 2, the joint density function of the multivariate distribution F is given by

$$f(X_1, X_2) = c(F_1(X_1), F_2(X_2))f_1(X_1)f_2(X_2), \tag{2}$$

where

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2),\ u_1, u_2 \in [0, 1], \tag{3}$$

denoting the copula density of $C(.,.)$, Joe and Xu (1996).

3.1 Archimedean copulas

Copulas are divided into numerous families. Having a large class of copulas at one's disposal is desirable for statistical modeling. The Archimedean copulas, found in the literature with one or more real parameters, constitute an important and extensive family of copulas. We utilize the Archimedean copula to model the dependence in WMOBE distribution due to:

1. The ease with which they can be constructed,
2. The great family of copulas which belongs to this class,
3. The nice properties possessed by this family of copulas.

A bivariate, exchangeable Archimedean copula is defined as

$$C(u_1, u_2) = \phi\{\phi^{-1}(u_1) + \phi^{-1}(u_2)\},\ u_1, u_2 \in [0, 1],$$

where $\phi \in L$, $\phi : [0, \infty] \to [0, 1]$ is called the generator of the copula and depends on θ, Härdle and Okhrin (2010). L denotes the class of Laplace transforms consisting of strictly decreasing, differentiable functions.

Archimedean copulas are widely used in applications because of their simple form, covering a variety of dependence structures and the nice properties they possess. Table (1) highlights the most important Archimedean copulas with their generators and parameter space.

Table 1: Most important Archimedean copulas.

Copula	Generator	Bivariate copula	Parameter
Clayton	$(1+\theta s)^{-\frac{1}{\theta}}$	$[\max\{u^{-\theta} + v^{-\theta} - 1; 0\}]^{-\frac{1}{\theta}}$	$\theta \in [-1, \infty)\backslash\{0\}$
Gumbel	$-(\log(s))^{\theta}$	$\exp[-((-\log(u))^{\theta} + ((-\log(v))^{\theta}))^{\frac{1}{\theta}}]$	$\theta \in [1, \infty)$
Frank	$-\log(\frac{\exp(-\theta s)-1}{\exp(-\theta)-1})$	$-\frac{1}{\theta}\log[1 + \frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1}]$	$\theta \in \mathbb{R}\backslash\{0\}$

According to (3), the PDF of the Archimedean copulas can be easily calculated. For example, in the case of the Clayton copula, the PDF is calculated as

$$c(u, v; \theta) = (\theta + 1)(uv)^{-(\theta+1)}(u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}-1},\ 0 \le u, v \le 1. \tag{4}$$

The joint PDF of the WMOBE distribution can take different shapes. We have presented the contour plots of WMOBE distribution with Archimedean copulas for $\theta = 1$

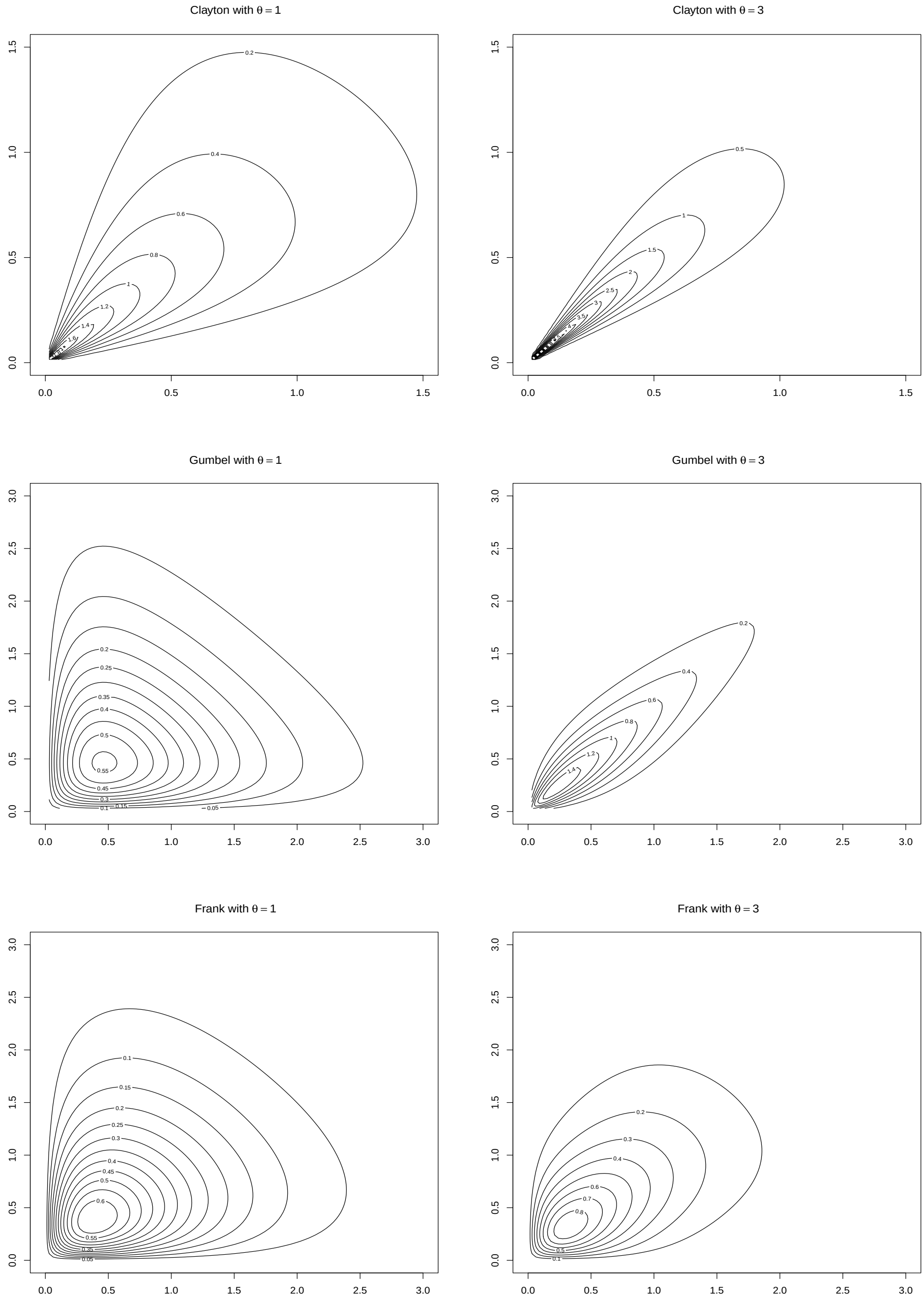

Figure 1: Contour plots of the joint PDF of the WMOBE distribution for $\alpha = 0.5\,, \lambda_1 = \lambda_2 = 1$ and $\lambda_{12} = 0.5$ with Clayton copula (first row), Gumbel copula (second row) and Frank copula (third row) for $\theta = 1, 3$.

and 3 in 1.This figure illustrates that the strength of dependence between the marginals intensifies with increasing θ.

In Archimedean copulas, measures of association play a crucial role in quantifying the strength of dependence between variables. Several measures of association are commonly used in the context of Archimedean copulas. Two important measures of association used more frequently are Spearman's rho (ρ) and Kendall's tau (τ). If X and Y are continuous random variables with copula C, then

$$\begin{aligned} \rho_{XY} &= 12\int_0^1\int_0^1 C(u,v)\mathrm{d}u\mathrm{d}v - 3, \\ \tau_{XY} &= 1-4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}\mathrm{d}u\mathrm{d}v. \end{aligned}$$

For Archimedean copulas, there is no simple expression for Spearman's ρ in terms of the generator ϕ, but for Kendall's τ, we have

$$\tau_{XY} = 1+4\int_0^1 \frac{\phi(t)}{\phi'(t)}\mathrm{d}t.$$

Table 2 represents the Kendalls τ for three Archimedean copulas Clayton, Frank and Gumbel, Weber (2015).

Table 2: Kendalls τ as a function of the copula parameter.

Archimedean copula	Kendallsτ
Clayton	$\frac{\theta}{\theta+2}$
Frank	$1+\frac{4}{\theta}[D_1(\theta)-1]^*$
Gumbel	$\frac{\theta-1}{\theta}$

$*D_k(x) = kx^{-k}\int_0^x t^k(e^t-1)^{-1}dt$ for $k=1,2.$

4 Estimation of copula models

Once the copula model has been selected, the next stage is to estimate the model. The maximum likelihood-based methods are the most common calibration methods. To assess the parameters of the copula model, we need to know the likelihood function. For this, consider the 2-dimensional copula model (2) under absolute continuity assumptions and a random sample size n of (i.i.d) vectors $\boldsymbol{x}_j = (x_{1j}, x_{2j}),\ j = 1,2,\ldots,n$. Suppose that the copula C belongs to a family of copulas indexed by a parameter $\theta : C(u_1,u_2;\theta)$. Also, let the marginal distribution functions F_i and its PDF f_i are indexed by (vector) parameters $\boldsymbol{\Lambda_i} : F_i(x_i;\boldsymbol{\Lambda_i})$ and $f_i(x_i;\boldsymbol{\Lambda_i}), i = 1,2,$ respectively. The likelihood function of (2) is defined as

$$L(\theta,\boldsymbol{\Lambda_1},\boldsymbol{\Lambda_2}) = \prod_{j=1}^{n} f(x_{1j},x_{2j};\theta,\boldsymbol{\Lambda_1},\boldsymbol{\Lambda_2}), \tag{5}$$

where

$$f(x_{1j},x_{2j};\theta,\mathbf{\Lambda_1},\mathbf{\Lambda_2}) = c(F_1(x_{1j};\mathbf{\Lambda_1}),F_2(x_{2j};\mathbf{\Lambda_2});\theta)f_1(x_{1j};\mathbf{\Lambda_1})f_2(x_{2j};\mathbf{\Lambda_2}). \quad (6)$$

With replacing (6) in (5), it is clear that the likelihood function is decomposed to likelihood contribution from dependence structure in data represented by the copula $c(.,.;\theta)$ and the likelihood contribution from margins, $f_i(x_i;\mathbf{\Lambda_i})$, $i=1,2$.

For estimating the parameters of copula models, we need the log-likelihood function of L, which can be written as below

$$\ell(\theta,\mathbf{\Lambda_1},\mathbf{\Lambda_2}) = \sum_{j=1}^{n} \log\, c(F_1(x_{1j};\mathbf{\Lambda_1}),F_2(x_{2j};\mathbf{\Lambda_2});\theta) + \sum_{i=1}^{2}\sum_{j=1}^{n} \log\, f_i(x_{ij};\mathbf{\Lambda_i}). \quad (7)$$

Now, we can find the ML estimates of parameters by maximizing the ℓ, numerically.

Lemma 4.1. *Let C is a bivariate Archimedean copula with parameter θ. Then for WMOBE* (2) *with parameter vector $(\theta,\mathbf{\Lambda_1},\mathbf{\Lambda_2})$ which $\mathbf{\Lambda_1}=(a_1,b_1)$ and $\mathbf{\Lambda_2}=(a_2,b_2)$, the log likelihood function can be written as below*

$$\begin{aligned}\ell\,(\theta,a_1,b_1,a_2,b_2) = {} & n\log(\frac{a_1+1}{a_1}) + n\log(b_1) - b_1\sum_{j=1}^{n}x_{1j} + \sum_{j=1}^{n}\log(1-e^{-a_1b_1x_{1j}}) \\ & +n\log(\frac{a_2+1}{a_2}) + n\log(b_2) - b_2\sum_{j=1}^{n}x_{2j} + \sum_{j=1}^{n}\log(1-e^{-a_2b_2x_{2j}}) \\ & +\sum_{j=1}^{n}\log\, c(1+\frac{1}{a_1}e^{-b_1(a_1+1)x_{1j}} - \frac{a_1+1}{a_1}e^{-b_1x_{1j}}, \\ & +\frac{1}{a_2}e^{-b_2(a_2+1)x_{2j}} - \frac{a_2+1}{a_2}e^{-b_2x_{2j}};\theta). \quad (8)\end{aligned}$$

The initial values of the parameters of marginal distributions which the (8) to be optimized over are calculated using the moment method as follows:

$$\hat{a}_i = \frac{2M_i-3+\sqrt{2M_i-3}}{2-M_i}\ ;\ \hat{b}_i = \frac{\hat{a}_i+2}{(\hat{a}_1+1)\bar{x}_1},$$

provided that $\frac{3}{2} < M_i < 2$ where $M_i = \frac{m_{2i}}{m_{1i}^2}$ such that m_{1i} and m_{2i} are the first and second order raw moments of $\boldsymbol{x}_i=(x_{i1},x_{i2},\ldots,x_{in}); i=1,2$. The initial value for the copula parameter is calculated using the Kendall's τ according to Table 2.

There are two likelihood-based methods to calibrate the model. The first one is full maximum likelihood (FML) which estimates the model parameters simultaneously. In this method, the estimates are consistent and asymptotically efficient. But in high dimensions of vector parameters, it might be computationally demanding. To alleviate this problem, Joe and Xu (1996) suggested an alternative to estimating parameters. This method is two-stage as follows

1. In the latter part of (7), the log likelihood functions of margins are optimized to estimate the parameters of the margins $\mathbf{\Lambda_i}\, i=1,2$. The $\mathbf{\Lambda_i}$'s are then replaced by

referred estimates in (7).
2. In the second step, the likelihood function (7) is maximized over θ,

$$\hat{\theta}_{IFM} = \text{Arg max } \ell(\theta; \hat{\mathbf{\Lambda}}_{\boldsymbol{i}}).$$

This method is referred to as Inference for Margins (IFM) and simplifies the computational burden of FML, Sakhaei and Nasiri (2020).

5 Simulation study

In this section, we assess the performance of the proposed estimation method utilizing a simulation study for different sample sizes in WMOBE distribution. The study is limited to the well-known Clayton copula for modeling dependence between margins. But for Frank and Gumbel copulas, the same results are obtained.

We first simulate sample paths of a WMOBE distribution with the rejection sampling method. The rejection sampling method is usually used to simulate unpopular distributions. Suppose we want to simulate data from bivariate PDF $f(x, y); x, y \in R$. In this method, we need a proposal PDF $g(x, y)$ that covers the support of the $f(x, y)$. $g(x, y)$ is easy to sample from and $\frac{f(x,y)}{g(x,y)} \leq c$, where $c \geq 1$ is a real number. The step-by-step procedure for the rejection sampling method is as follows
1. let (x, y) be a random variable with PDF $f(x, y)$ $\forall x, y \in R$,
2. let (x', y') be a simulated random variable with PDF $g(x, y)$ $\forall x, y \in R$,
3. let $\frac{f(x',y')}{g(x',y')} \leq c$ $\forall x', y' \in R, c \geq 1$ is a real number,
4. let $0 < R_1 < 1$, $0 < R_2 < 1$ be two random numbers,
5. set x' in terms of R_1 and y' in terms of R_2 depending on the expression obtained for the ratio $\frac{f(x',y')}{kg(x',y')}$,
6. if $R_1 R_2 \leq \frac{f(x',y')}{kg(x',y')}$, then set $(x, y) = (x', y')$ else reject the (x', y') and repeat the process from (1).

Table 3 shows the results of the parameter's estimation of WMOBE distribution with IFM procedure with corresponding standard errors. It is observed that with increasing n, the standard errors of estimates are reduced. Additionally, for each n, the standard error of the $\hat{\theta}$ increases as θ increases. Figure 2 shows the boxplot of marginal parameter estimates for sample sizes $n = 100, 500, 1000$. The true values of the marginal parameters are $a_1 = 1, b_1 = 1.5, a_2 = 1$ and $b_2 = 1.5$ and are indicated by horizontal lines. Since the copula parameter is estimated separately in the second step of the IFM method, its boxplot is shown in Figure 3 for different sample sizes. The number of replicates in all simulations is equal to 20. It is observed that the IFM estimation procedure slightly overestimates or underestimates the copula parameter for small sample sizes. Therefore, increasing the sample sizes shows better effects. In addition, the copula parameter estimates are more accurate for large sample sizes than small ones.

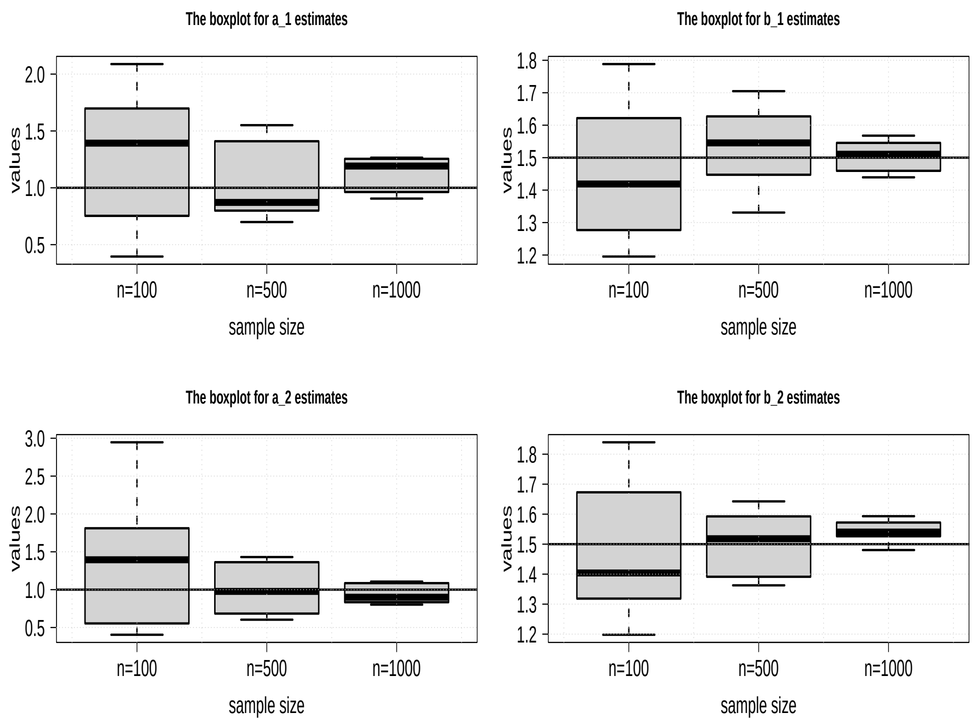

Figure 2: The boxplot of marginal parameter estimations with true values $a_1 = 1$ (top left), $b_1 = 1.5$ (top right), $a_2 = 1$ (below left) and $b_2 = 1.5$ (below right) for simulated data with sample sizes $n = 200, 500, 1000$.

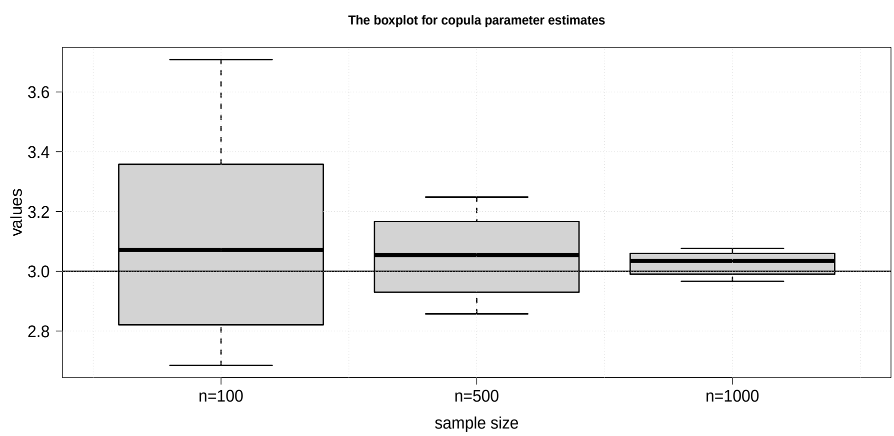

Figure 3: The boxplot of Clayton copula parameter estimate ($\hat{\theta}$) for simulated data with different values of n and $\theta = 3$.

Table 3: IFM estimates of parameter values and the corresponding root mean squared errors.

n	True values					IFM estimates				
	a_1	b_1	a_2	b_2	θ	$\hat{a}_1$	$\hat{b}_1$	$\hat{a}_2$	$\hat{b}_2$	$\hat{\theta}$
100	1	1.5	1	1.5	3	0.4885(2.0278)	1.8234(1.0094)	2.1647(1.9559)	1.4862(0.2611)	3.4525(0.3936)
	1	2	1	2	3	3.4346(2.0987)	1.8646(0.2424)	3.6974(1.9638)	1.6810(0.2043)	3.4177(0.3954)
	1.25	2	1.25	2	5	1.3046(1.6685)	1.9694(0.4668)	1.7053(1.6400)	1.8421(0.3462)	4.9765(0.5245)
	1	2.5	1	2.5	5	0.6666(1.2529)	2.6478(0.7795)	2.3155(1.3429)	1.9822(0.2640)	5.5563(0.5679)
200	1	1.5	1	1.5	3	1.8395(1.0610)	1.3630(0.1601)	0.3198(1.0986)	1.7776(0.6456)	3.5557(0.2841)
	1	2	1	2	3	1.8995(1.2838)	1.7978(0.2360)	3.6059(1.7118)	1.6227(0.1546)	2.9171(0.2472)
	1.25	2	1.25	2	5	0.8712(0.7812)	2.2798(0.3600)	1.3692(0.8017)	2.1185(0.2525)	5.1092(0.3806)
	1	2.5	1	2.5	5	1.4304(0.9537)	2.2461(0.2949)	1.2709(0.9457)	2.2820(0.3249)	6.0229(0.4296)
400	1	1.5	1	1.5	3	1.3366(0.5919)	1.4479(0.1280)	0.7647(0.7354)	1.5853(0.2485)	2.9213(0.1757)
	1	2	1	2	3	1.1619(0.6671)	1.9818(0.2127)	0.6487(0.7368)	2.2286(0.3877)	2.8729(0.1755)
	1.25	2	1.25	2	5	1.4212(0.6884)	1.9266(0.1826)	1.5172(0.7600)	1.9156(0.1864)	5.2558(0.2713)
	1	2.5	1	2.5	5	0.5567(0.6365)	2.6613(0.4398)	0.3498(0.6963)	2.8764(0.6418)	4.8652(0.2568)
600	1	1.5	1	1.5	3	0.6273(0.5245)	1.6780(0.2138)	1.3600(0.5673)	1.4165(0.1141)	2.9889(0.1464)
	1	2	1	2	3	0.5575(0.6052)	2.1123(0.3288)	0.7573(0.5628)	2.0144(0.2444)	2.8204(0.1406)
	1.25	2	1.25	2	5	1.4048(0.5100)	1.9166(0.1390)	1.0586(0.5681)	2.0275(0.1971)	5.0766(0.2167)
	1	2.5	1	2.5	5	0.9143(0.4329)	2.4325(0.2078)	0.5251(0.4475)	2.6892(0.3254)	5.0132(0.2142)
1000	1	1.5	1	1.5	3	1.4202(0.4476)	1.4455(0.0886)	1.9510(0.4888)	1.3340(0.0684)	3.0580(0.1150)
	1	2	1	2	3	1.2937(0.4307)	1.9270(0.1229)	0.6079(0.5079)	2.1880(0.1713)	3.1076(0.1155)
	1.25	2	1.25	2	5	1.1368(0.4518)	2.0339(0.1486)	1.4267(0.4045)	1.9194(0.1085)	4.9668(0.1639)
	1	2.5	1	2.5	5	0.5721(0.4528)	2.5432(0.2926)	0.2480(0.5658)	2.8048(0.5698)	4.4490(0.1502)
2000	1	1.5	1	1.5	3	1.0447(0.2987)	1.4613(0.0759)	1.1786(0.2832)	1.4403(0.0656)	3.0397(0.0810)
	1	2	1	2	3	1.4238(0.3449)	1.8921(0.0876)	1.5105(0.3136)	1.8889(0.0775)	3.0043(0.0797)
	1.25	2	1.25	2	5	0.7725(0.2556)	2.1839(0.1211)	1.0903(0.2769)	1.9956(0.0943)	4.8942(0.1155)
	1	2.5	1	2.5	5	0.6900(0.2799)	2.5446(0.1640)	0.9039(0.2757)	2.4263(0.1299)	4.7621(0.1122)

6 Real data analysis

In this section, our objective is to apply the proposed model to real data. For this, we focus on studying automobile insurance data. There is little evidence of studying automobile insurance data that are naturally asymmetric in practice due to large claims. The data is collected from the ASIA Insurance (an IRANIAN insurance company) monthly reports to analyze the risk premium from 2018 to 2020, comprising 623 observations. The data consists of the number of people killed or injured in accidents (PK/I) and the number of claims in Third-Party contracts filed (TPC). We used the log return of the data. A summary statistics of the data is given in Table 4. Sample Kurtosis of 7.7840 and 4.7623 for PK/I and TPC, respectively, suggest a weighted exponential distribution for both margins.

Table 4: Summary Statistics of PK/I and TPC Variables.

Variables	Mean	Variance	Skewness	Kurtosis	Minimum	Maximum
PK/I	0.5769	0.1987	1.7813	7.7840	0.0062	3.0083
TPC	0.6159	0.1935	1.2241	4.7623	0.0046	2.6710

We can find the maximum likelihood estimation (MLE) parameter of marginal distributions by maximizing the first and second line of (8) numerically. So, the fitted weighted exponential distributions are as follows

$$\begin{aligned} f_X(x) &= \frac{1.4216+1}{1.4216} 2.4494 e^{-2.4494x}(1-e^{-(1.4216)(2.4494)x}),\ x>0, \\ f_Y(y) &= \frac{0.4026+1}{0.4026} 2.7812 e^{-2.7812y}(1-e^{-(0.4026)(2.7812)y}),\ y>0, \end{aligned}$$

The Q-Q plot of PK/I and TPC variables are depicted in Figure 4. The x-axis represents the quantiles of the theoretical distribution, while the y-axis represents the quantiles of the observed data. As observed, there are no departures from the expected distributions, providing insights into the nature of the data's distributional characteristics.

In order to assessing the goodness of fit of a statistical model to a set of data, the Cramér-von Mises criteria is used. It measures the discrepancy between the empirical distribution function of the observed data and the theoretical distribution function proposed by the model. The formula for the Cramér-von Mises statistic is

$$W^2 = \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \, dF(x),$$

where W^2 is the Cramér-von Mises statistic, $F_n(x)$ is the empirical cumulative distribution function of the observed data and $F(x)$ is the cumulative distribution function of the theoretical distribution being tested. More Specifically, $W^2 = 0.6973(Sig = 0.3547)$ and $W^2 = 0.7342(Sig = 0.2071)$ indicates that there is no significant discrepancy between the observed data and the theoretical distribution WMOBE for PK/I and TPC, respectively.

Figure 5 shows the plot of empirical cumulative distribution function(ECDF) and the theoretical cumulative distribution function (TCDF) for two studying variables.

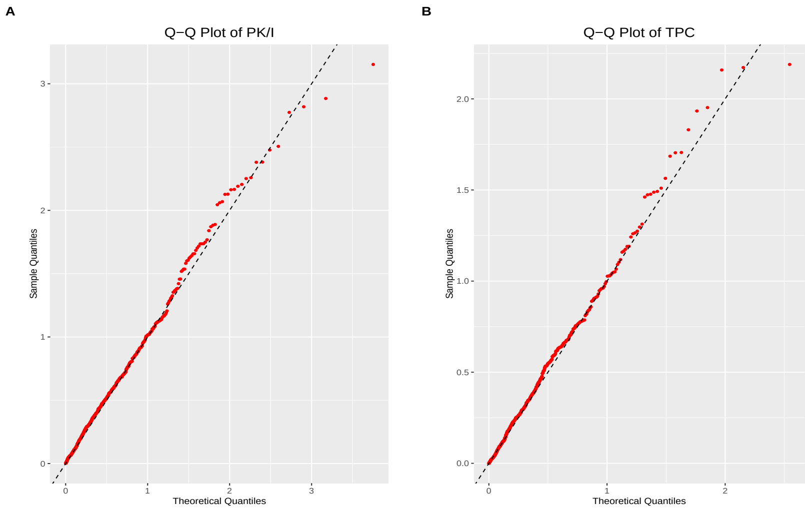

Figure 4: Q-Q plot of PK/I (left) and TPC (right) variables for real data.

The proximity of these two functions implies that the weighted exponential distribution is a good proposal for the distribution of the studying variables.

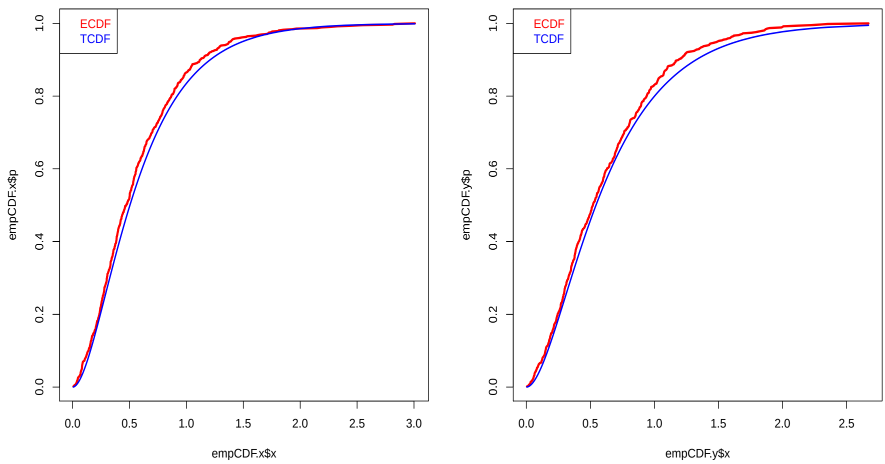

Figure 5: ECDF (empirical CDF) and TCDF (theoretical CDF) of PK/I(left) and TPC(right) variables.

Figure 6 shows the scatterplot of PK/I values versus TPC values. From this Figure, it appears that there exists a strong level of dependence between variables which is supported by the associated Kendall's $\tau = 0.6126$(sig=$2.2e-16$). The closer this coefficient is to 1, it tells us that the variables are dependent on each other.

For model selection among a set of candidate models, the AIC (Akaike information

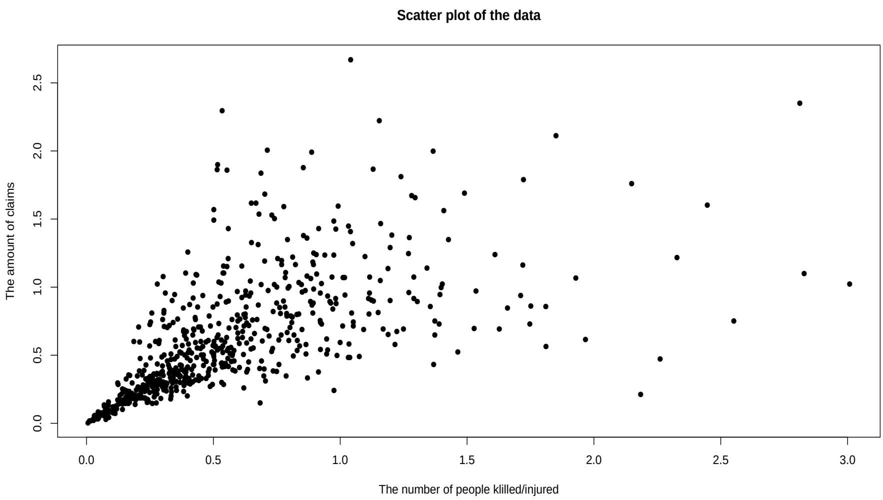

Figure 6: Scatter plot of PK/I vs TPC data.

criterion) and BIC (Bayesian information criterion) are used. Both of them balance the trade-off between goodness of fit and model complexity. The model with the lowest AIC and BIC value is preferred, indicating that it provides the best candidate. Table 5 represents the results of using the above diagnostic criteria along with copula parameter estimates corresponding to the Archimedean bivariate copula family and the model with no copula which is considered by Jamalizadeh and Kundu (2013). The WMOBE model with Clayton copula represents the best fit for the pair of variables that were being tested.

Table 5: The comparison of WMOBE model between Archimeadn copula and non-copula model.

Model	$\hat{\theta}$	AIC	BIC	log likelihood
Clayton Copula	3.0187	-842.3132	-837.8786	422.1566
Frank Copula	8.1263	-622.2713	-617.8368	312.1357
Gumbel Copula	2.1031	-484.7729	-480.3383	243.3864
Non-copula	-	-158.04212	-140.6830	83.2106

7 Conclusion

This paper used the copula approach to characterize the dependence structure between margins in weighted Marshall-Olkin bivariate exponential distribution. Through a simulation study, the behavior of copula parameter estimates is investigated using the IFM method. It is observed that the IFM method tends to overestimate the copula parameter for small sample sizes. However, as the sample size increases, the standard errors of the estimates decrease. Furthermore, keeping n constant, an increasing in θ leads to a reduction in the dispersion of the copula parameter estimate.

The model's parameters were estimated by applying them to real insurance data. A comparison between the theoretical CDF and the empirical CDF of the data indicates that the marginal distributions of the desired model align well with the weighted exponential distribution. Diagnostic criteria results reveal that the Clayton copula in Archimedean copula families is suitable for describing the dependence structure.

Acknowledgement

The authors would like to acknowledge reviewer(s) for their valuable comments and technical assistance, which have improved the quality of this paper.

References

Al-Mutairi, D.K., Ghitanya, M.E. and Kundu, D. (2018). Weighted Weibull distribution: Bivariate and multivariate cases. *Brazilian Journal of Probability and Statistics*, **32**(1):20–43.

Arnold, B.C. and Beaver, R.J. (2000). Hidden truncation models. *Sankhyā: The Indian Journal of Statistics, Series A*, **62**(1):23–35.

Arnold, B.C., Beaver, R.J., Azzalini, A., Balakrishnan, N., Bhaumik, A., Dey, D.K., Cuadras, C.M., Sarabia, J.M., Arnold, B.C. and Beaver, R.J. (2002). Skewed multivariate models related to hidden truncation and/or selective reporting. *TEST: An Official Journal of the Spanish Society of Statistics and Operations Research,* **11**:7–54.

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, **12**(1):171–178.

Cherubini, U., Luciano, E. and Vecchiato, W. (2004). *Copula Methods in Finance*. John Wiley & Sons.

Fallah, A. and Kazemi, R. (2022). Statistical inference for the generalized weighted exponential distribution. *Communications in Statistics-Simulation and Computation*, **51**(8):4296–4313.

Genest, C., Gendron, M. and Bourdeau-Brien, M. (2013). The advent of copulas in finance. *The European Journal of Finance*, **15**(7-8):609–618.

Gupta, R.D. and Kundu, D. (2009). A new class of weighted exponential distributions. *Statistics*, **43**(6):621–634.

Härdle, W.K. and Okhrin, O. (2010). De copulis non est disputandum: Copulae: an overview. *AStA Advances in Statistical Analysis*, **94**(1):1–31.

Iqbal, T. and Iqbal, M.Z. (2020). On the mixture of weighted exponential and weighted Gamma distribution. *International Journal of Analysis and Applications*, **18**(3):396–408.

Jamalizadeh, A. and Kundu, D. (2013). Weighted Marshall-Olkin bivariate exponential distribution. *Statistics*, **47**(5):917–928.

Joe, H. and Xu, J.J. (1996). The estimation method of inference functions for margins for multivariate models. *Technical Report*, No. 166, Department of Statistics, University of British Columbia, Vancouver. https://dx.doi.org/10.14288/1.0225985

Kazemi, R., Kohansal, A., Fallah, A. and Nasiri, F. (2021). Inference on a four-parameter generalized weighted exponential distribution. *Filomat*, **35**(3):827–844.

Khan, M.J.S. and Kumar, S. (2016). Concomitant of order statistics from weighted Marshall-Olkin bivariate exponential distribution. *IOSR Journal of Mathematics*, **12**(6):99–108.

Kotz, S., Balakrishnan, N. and Johnson, N.L. (2019). *Continuous Multivariate Distributions, Volume 1: Models and Applications*. Vol. 334, John Wiley & Sons.

Mahdavi, A., Fathizadeh, M. and Jamalizadeh, A. (2017). On the bivariate weighted exponential distribution based on the generalized exponential distribution. *Communications in Statistics-Theory and Methods*, **46**(8):3641–3648.

Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, **84**(3):641–652.

Riad, F.H., Hussam, E., Gemeay, A.M., Aldallal, R.A. and Afify, A.Z. (2022). Classical and Bayesian inference of the weighted-exponential distribution with an application to insurance data. *Mathematical Biosciences and Engineering*, **19**(7):6551–6581.

Rubio, F.J. and Steel, M.F. (2012). On the Marshall-Olkin transformation as a skewing mechanism. *Computational Statistics & Data Analysis*, **56**(7):2251–2257.

Sakhaei, A. and Nasiri, P. (2020). Estimating the parameters of periodic bivariate compound Poisson process by inference for margins method. *Journal of Statistical Sciences*, **13**(2):461–482.

Shahbaz, S., Shahbaz, M.Q. and Butt, N.S. (2010). A class of weighted Weibull distribution. *Pakistan Journal of Statistics and Operation Research*, **6**(1):53–59.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Annales de l'ISUP*, **8**(3):229–231.

Tapan, C. (2022). The new discrete weighted exponential distribution and application. *The Journal of Applied Science*, **21**(2):244452.

Weber, V. (2015). *Estimation of The Dependence Parameter in Bivariate Archimedean Copula Models Under Misspecification*. Master's thesis, Humboldt-Universität zu Berlin, Wirtschaftswissenschaftliche Fakultät.