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Orderings of fail-safe systems with distribution-free components under random shocks

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Abstract: Often, reliability systems suffer shocks from external stress factors, stressing the system at random. These random shocks may have non-ignorable effects on the reliability of the system. In this paper, we provide sufficient and necessary conditions on components' lifetimes and their survival probabilities from random shocks for comparing the lifetimes of two fail-safe systems in two cases: (i) when components are independent, and then (ii) when components are dependent. We then apply the results for some distribution-free random variables with possibly different parameters to illustrate the established results.

Keywords: Archimedean copula; Distribution-free random variables; Fail-safe systems; Random shocks; Stochastic orders.
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1 Introduction

One of the most commonly used systems in reliability is an r-out-of-n system. This system, comprising n components, works iff at least r components work, and it includes parallel, fail-safe and series systems all as special cases corresponding to $r = 1$, $r = n-1$ and $r = n$, respectively. Let $X_1, \ldots, X_n$ denote the lifetimes of components of a system and $X_{1:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Then, $X_{n-r+1:n}$ corresponds to the lifetime of a r-out-of-n system. Due to this direct connection, the theory of order statistics becomes important in studying $(n-r+1)$-out-of-n systems and

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in characterizing their properties. For comprehensive discussions on the study of order statistics and their applications, one may refer to Balakrishnan and Rao (1998a,b).

The comparison of important characteristics associated with lifetimes of technical systems is an interesting topic in reliability theory, since it usually enables us to approximate complex systems with simpler ones and subsequently enable us to obtain various bounds for important ageing characteristics of the complex system. A tool that is useful for this purpose is the theory of stochastic orderings.

Let $X_1, \ldots, X_n$ be non-negative independent random variables corresponding to the lifetimes components of a system. Let $I_{p_1}, \ldots, I_{p_n}$, independent X_i's, be independent Bernoulli random variables such that $E[I_{p_i}] = p_i$, $i = 1, \ldots, n$, with $I_{p_i} = 1$ if component i survives from the random shock, otherwise $I_{p_i} = 0$ if component i fails due to the shock, for $i = 1, \ldots, n$. For a given time period, we can use $I_{p_1}X_1, \ldots, I_{p_n}X_n$ to denote the components' lifetimes subject to random shocks. Of special interest are $Y_{n:n} = \max(I_{p_1}X_1, \ldots, I_{p_n}X_n)$ and $Y_{1:n} = \min(I_{p_1}X_1, \ldots, I_{p_n}X_n)$ corresponding to the lifetime of parallel and series systems, respectively. Throughout, the term "heterogeneity" among components indicates diversity among them, which is then compared by majorization order imposed on the distribution parameters with corresponding systems with homogeneous components with similar assumptions being made on the survival probabilities. It is of course interesting to look into the influence of heterogeneity among the components and the random shocks on the lifetimes of parallel and series systems.

We can also present an alternative version of the problem stated above as follows. Consider systems with a finite number of components, each of which is equipped with a starter whose performance is modelled by a Bernoulli random variable, with all component lifetimes being independent. Since each starter may fail to initiate the component, the total number of components in operation will be a random number. Such situations arise in some practical applications. For example, the reliability and availability of power plants typically begins with the gas turbine start-up procedure, the time length of an on-line conference is the maximum on-line time of those who successfully registered for the conference, and the largest loss of an insured with a policy covering multiple risks is the maxima of those invoked losses only. Another scenario in auction theory is as follows: an auctioneer usually attracts some potential bidders through advertising a precious object, and in this case the largest bid of those participants defines the the price of the object for sale; see Li (2005); Fang and Li (2015); Li and Li (2019).

In actuarial science, X_i's may represent claim sizes of risks covered by one policy and I_{p_i}'s indicate the occurrence of these claims. Then, $Y_{n:n} = \max(I_{p_1}X_1, \ldots, I_{p_n}X_n)$ and $Y_{1:n} = \min(I_{p_1}X_1, \ldots, I_{p_n}X_n)$ correspond to the largest and smallest claim amounts in a portfolio of risks, respectively.

Fail-safe systems are commonly used in many day-to-day applied structures, because of its fault tolerance or failure-survivability design. A fail-safe is a special design feature that, when a failure occurs, will respond in such a way that no harm happens to the system itself. The brake system in a train is a good example of a fail-safe system in which the brakes are held in off-position by air pressure and if a brake line splits or a carriage becomes separated, the air pressure will be lost and in that case the brakes will get applied by a local air reservoir. Yet another classic example of a fail-safe system is an elevator in which brakes are held off brake pads by tension and if the tension gets lost, the brakes latch on the rails in shaft thus preventing the elevator from falling off. There are many other such fail-safe systems in common use.

Balakrishnan et al. (2015) established necessary and sufficient conditions for comparing two fail-safe systems with independent homogeneous exponential components, in the sense of mean residual life, dispersive, hazard rate and likelihood ratio orders. Their results specifically showed how an $(n-1)$-out-of-n system consisting of heterogeneous components with exponential lifetimes can be compared with any $(m-1)$-out-of-m system consisting of homogeneous components with exponential lifetimes. In a similar vein, Zhang et al. (2019) presented sufficient (and necessary) conditions on lifetimes of components and their survival probabilities from random shocks for comparing the lifetimes of two fail-safe systems in terms of usual stochastic, hazard rate and likelihood ratio orders.

Cai et al. (2017) compared the hazard rate order of second order statistics arising from two sets of independent multiple-outlier proportional hazard rates (PHR) samples. They then established the submajorization order between the sample size vectors together with the supermajorization order between the hazard rate vectors imply the hazard rate ordering between the corresponding second-order statistics from multiple-outlier PHR random variables.

Zhang et al. (2019) have discussed the usual stochastic, hazard rate, and likelihood ratio orderings between two fail-safe systems comprising independent components subject to independent random shocks. These results can be also applied to analyze the effects of heterogeneity among pricing distribution and attending probabilities on the actual cost of the auctioneer in the context of second-price reverse auction.

Barmalzan et al. (2022) established stochastic comparisons of two fail-safe systems with dependent components having an Archimedean copula for the joint survival function and general exponentiated location-scale lifetime distributions. Recently, Hazra et al. (2022) obtained some comparisons results for the second smallest and the second largest order statistics from a general semiparametric family of distributions using different stochastic orders, namely, the usual stochastic order, the hazard rate order and the reversed hazard rate order.

Most of the existing literature on stochastic comparisons of fail-safe systems have focused on the case when the components in the systems are all independent. However, when such technical systems are in operation, many important factors, such as operating conditions, environmental conditions and stress factors, are shared and experienced by all the components in the system. It would, therefore, be reasonable for the lifetimes of components in a system to be dependent. Of course, there are many ways to model this dependence (Kotz et al., 2000), and the theory of copulas is one useful tool for this purpose; see, for example, Nelsen (2006) for a book length account of copulas. Though many copulas have been developed in the literature, Archimedean copulas have been studied extensively by many researchers due to their flexibility and also due to the fact that they include the well-known Clayton copula, Ali-Mikhail-Haq copula, and Gumbel-Hougaard copula all as special cases. Moreover, they also include the independence copula as a special case and consequently comparison results established under Archimedean copulas for the joint distribution of lifetimes of components in the system are quite general and would naturally include the corresponding results for the case of independent components.

In the present work, from a reliability viewpoint, we consider the lifetimes of fail-safe systems (i.e., the second order statistics) in a general scenario when (i) the component lifetimes are independent and (ii) the component lifetimes are dependent with their joint survival function represented by an Archimedean copula and having distribution-free lifetimes.

Now, let us give a reliability explanation of the our established results. Consider a factory that produces some specific units with fail-safe structures made up n components. Suppose the components used in building the units suffer shocks from external stress factors and also come from a supplier, say Supplier I. Supplier I asserts the lifetimes of its produced components follow the arbitrary model with possibly different parameters. For some reasons such as high price or unavailability of the components in a specific period of time, the factory decides to purchase its required components from a new supplier, say Supplier II. In the produced components by Supplier II, suppose the components used in building the units also suffer shocks from external stress factors and their lifetimes follow the another arbitrary models with possibly different parameters. In such a case, changing the components may impress the quality of the units of the factory. Therefore, to avoid the quality loss of the units, the factory must investigate the effect of these changes. In this situation, our results give some sufficient conditions to compare the survival functions of the units comprising the components of Suppliers I and II.

Another possible interpretation of our results established here in the area of auction theory can be stated as follows: In second-price reverse auction, bidders (sellers) submit sealed bids to the auctioneer (buyer) who solicits the purchase of items when the auction begins. The lowest bidder wins the bid and will be paid the amount of the second lowest price from the auctioneer. This type of auction is often used by large corporations and government departments to purchase supplies and services. The cost of the auctioneer can be expressed as $X_{2:n}$ if there are n bidders submitting prices $X_1, \ldots, X_n$. It may happen that some of the bidders drop out of the auction before the beginning due to some unforeseen circumstances. As a result, the final cost on the auction turns out to be the second-order statistics arising from $I_{p_1}X_1, \ldots, I_{p_n}X_n$ where I_{p_i} denotes whether bidder i attends the auction or not. Therefore, all the results in this paper can be used to explain the effects of the heterogeneity among the price distribution and attending probabilities on the actual cost of the auctioneer.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and notation pertinent to stochastic orders, vector majorization and related orders and also Archimedean copulas. Section 3 discusses stochastic comparisons of fail-safe systems under random shocks, with independent heterogeneous components, by using the concept of vector majorization between vectors of parameters. In Section 4, we establish some results on fail-safe systems under random shocks, when components are dependent. Finally, some concluding remarks are made in Section 5.

2 Preliminaries

We present here some basic definitions and lemmas that are used subsequently in establishing the main results. Throughout this paper, we denote $R_+ = (0, +\infty)$. In addition, we use $a \overset{sgn}{=} b$ to denote that both sides of an equality have the same sign.

Definition 2.1. *Suppose X and Y are two non-negative continuous random variables with distribution functions F_X and F_Y, survival functions $\bar{F}_X$ and $\bar{F}_Y$, respectively. We assume that all expectations exist wherever they are given. Then, X is said to be larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$ for all $t \in \mathbb{R}_+$, which is equivalent to saying that $\mathbb{E}(\phi(X)) \geq \mathbb{E}(\phi(Y))$ for all increasing functions $\phi : \mathbb{R}_+ \to \mathbb{R}$.*

Interested readers may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for comprehensive discussions on various stochastic orderings and inter-relationships between them.

Definition 2.2. *Consider two vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ with corresponding increasing arrangements $a_{1:n} \leq \ldots \leq a_{n:n}$ and $b_{1:n} \leq \ldots \leq b_{n:n}$. Then,*
(i) $\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{m}{\succ} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{j:n} \leq \sum_{j=1}^{i} b_{j:n}$, for $i = 1, \ldots, n-1$, and $\sum_{j=1}^{n} a_{j:n} = \sum_{j=1}^{n} b_{j:n}$;
(ii) $\boldsymbol{a}$ is said to weakly supermajorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{j:n} \leq \sum_{j=1}^{i} b_{j:n}$, for $i = 1, \ldots, n$.

The concept of majorization is a way of comparing two vectors of same dimension, in terms of dispersion of their components for which the order $\boldsymbol{u} \overset{m}{\succeq} \boldsymbol{v}$ means that u_i's are more dispersed than v_i's, for a fixed sum. For example, we always have $\boldsymbol{u} \overset{m}{\succeq} \bar{\boldsymbol{u}}$, where $\bar{\boldsymbol{u}} = (\bar{u}, \ldots, \bar{u})$ with $\bar{u} = n^{-1} \sum_{i=1}^{n} u_i$. It is evident that the majorization order implies both weak supermajorization and weak submajorization orders.

Definition 2.3. *We say that a real-valued function ϕ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$, is Schur-convex on $\mathbb{A}$ if*

$$\boldsymbol{u} \overset{m}{\succeq} \boldsymbol{v} \Rightarrow \phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v}) \qquad \text{for any } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{A}.$$

ϕ is said to be Schur-concave function on $\mathbb{A}$ if $-\phi$ is Schur-convex on $\mathbb{A}$.

Lemma 2.4. *(Marshall et al. (2011), p. 84) Suppose $I \subset \mathbb{R}$ is an open interval and $\Psi : I^n \to \mathbb{R}$ is continuously differentiable. Then, ψ is Schur-convex (Schur-concave) on I^n if and only if*
(i) Ψ is symmetric on I^n,
(ii) for all $i \neq j$ and all $\boldsymbol{z} \in I^n$,

$$(z_i - z_j)\left(\frac{\partial \Psi}{\partial z_i}(\boldsymbol{z}) - \frac{\partial \Psi}{\partial z_j}(\boldsymbol{z})\right) \geq 0 (\leq 0),$$

where $\frac{\partial \Psi}{\partial z_i}(\boldsymbol{z})$ denotes the partial derivative of Ψ with respect to its i-th argument.

Lemma 2.5. *(Marshall et al. (2011), p. 87) Consider the real-valued function ϕ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. Then, $\boldsymbol{u} \overset{w}{\succeq} \boldsymbol{v}$ implies $\phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v})$ if and only if ϕ is decreasing and Schur-convex on $\mathbb{A}$.*

Archimedean copulas have been widely used in reliability theory and actuarial science due to its mathematical tractability as well as its capability of capture wide ranges of dependence. For a decreasing and continuous function $\psi : [0, \infty) \longrightarrow [0, 1]$ such that $\psi(0) = 1$ and $\psi(+\infty) = 0$ and $\phi = \psi^{-1}$ being the inverse,

$$C_\psi(u_1, \ldots, u_n) = \psi(\phi(u_1) + \ldots + \phi(u_n)) \qquad \text{for all } u_i \in [0, 1], \;\; i = 1, \ldots, n,$$

is called an Archimedean copula with generator ψ if $(-1)^k \psi^{[k]}(x) \geq 0$ for $k = 0, \ldots, n-2$ and $(-1)^{n-2}\psi^{[n-2]}(x)$ is decreasing and convex. The Archimedean copula family includes many knows copulas, including the well-known independence (product) copula, Clayton copula, Ali-Mikhail-Haq (AMH) copula, Hougaard copula and Gumbel-Hougaard copula. For more discussions on copulas and their properties, one may refer to Nelsen (2006) and McNeil and Neslehova (2009).

3 Stochastic comparisons with independent components

This section, using the concepts of vector majorization, weakly supermajorization and related orders, presents stochastic comparisons of fail-safe systems in the sense of usual stochastic order when distribution-free components are independent.

Zhang et al. (2019) discussed stochastic comparison of fail-safe systems comprising independent components subject to independent random shocks and obtained results under the following set:

$$\mathcal{E}_n = \left\{ (\boldsymbol{a}, \boldsymbol{b}) = \left(\begin{array}{c} a_1 \ldots a_n \\ b_1 \ldots b_n \end{array} \right) : \; a_i, b_j > 0 \;\; and \;\; (a_i - a_j)(b_i - b_j) \geq 0, \;\; i, j = 1, \ldots, n \right\}.$$

In the throughout the paper, we consider the following sets

$$\begin{aligned} \mathcal{E}_n^+ &= \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n > 0\} \\ \mathcal{D}_n^+ &= \{(x_1, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \ldots \leq x_n\} \end{aligned}$$

In the following theorem, we suppose that underlying independent random variables have two parameters.

Theorem 3.1. *Suppose $X_1, \ldots, X_n$ are independent nonnegative random variables with $X_i \sim \bar{F}(\cdot; \alpha_i, \beta_i)$, where $\bar{F}(\cdot; \alpha_i, \beta_i)$ denotes the survival function of X_i, and $\alpha_i > 0$ and $\beta_i > 0$ are the distribution parameter of X_i, for $i = 1, 2, \ldots, n$. Let $I_{p_1}, \ldots, I_{p_n} (I_{p_1^*}, \ldots, I_{p_n^*})$ be a set of independent Bernoulli random variables, independent of X_i's, with $\mathbb{E}[I_{p_i}] = p_i (\mathbb{E}[I_{p_i^*}] = p_i^*), i = 1, 2, \ldots, n$. Let $Y_i = I_{p_i} X_i$ and $Y_i^* = I_{p_i^*} X_i, i = 1, 2, \ldots, n$. Assume that the following conditions hold:*
(i) $g : [0, 1] \longmapsto \mathbb{R}_+$ is a differentiable and strictly decreasing function;
(ii) $g^{-1}(u)$ is log-convex in $u \in \mathbb{R}_+$;
(iii) $\bar{F}(\cdot; \alpha, \beta)$ is decreasing in $\alpha \in \mathbb{R}_+$;
(iv) $\bar{F}(\cdot; \alpha, \beta)$ is increasing in $\beta \in \mathbb{R}_+$;
Then, for $\boldsymbol{g(p)}, \boldsymbol{g(p^*)}, \boldsymbol{\alpha} \in \mathcal{E}_n^+$ and $\boldsymbol{\beta} \in \mathcal{D}_n^+$ (or, $\boldsymbol{g(p)}, \boldsymbol{g(p^*)}, \boldsymbol{\alpha} \in \mathcal{D}_n^+$ and $\boldsymbol{\beta} \in \mathcal{E}_n^+$), we have

$$\big(g(p_1), g(p_2), \ldots, g(p_n)\big) \stackrel{w}{\succeq} \big(g(p_1^*), g(p_2^*), \ldots, g(p_n^*)\big) \Longrightarrow Y_{2:n} \geq_{st} Y_{2:n}^*.$$

Proof. Here, we present the proof only for the case when $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, $g(p_1) \geq g(p_2) \geq \ldots \geq g(p_n)$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, since the proof for the other case is quite similar. Let us set $g(p_i) = u_i$ for $i = 1, 2, \ldots, n$ and g^{-1} denotes the inverse function of g. The survival function of $Y_{2:n}$, for $t \geq 0$, can be expressed as

$$\begin{aligned}
\bar{F}_{Y_{2:n}}(t) &= \mathbb{P}(Y_{2:n} > t) \\
&= \sum_{i=1}^{n} \mathbb{P}(Y_i \leq t, Y_j > t, for\, j \neq i) + \mathbb{P}(Y_1 > t, \ldots, Y_n > t) \\
&= \sum_{i=1}^{n} \left[F_{Y_i}(t) \prod_{j \neq i}^{n} \bar{F}_{Y_j}(t) \right] + \prod_{i=1}^{n} \bar{F}_{Y_i}(t) \\
&= \sum_{i=1}^{n} \left[(1 - p_i \bar{F}(t; \alpha_i, \beta_i)) \prod_{j \neq i}^{n} p_j \bar{F}(t; \alpha_j, \beta_j) \right] + \prod_{i=1}^{n} p_i \bar{F}(t; \alpha_i, \beta_i) \\
&= \sum_{i=1}^{n} \prod_{j \neq i}^{n} p_j \bar{F}(t; \alpha_j, \beta_j) - (n-1) \prod_{i=1}^{n} p_i \bar{F}(t; \alpha_i, \beta_i) \\
&= \sum_{i=1}^{n} \prod_{j \neq i}^{n} g^{-1}(u_j) \bar{F}(t; \alpha_j, \beta_j) - (n-1) \prod_{i=1}^{n} g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i).
\end{aligned}$$

To obtain the desired result, according to Lemma 2.5, it suffices to show that for each fixed $t \geq 0$, $\bar{F}_{Y_{2:n}}(t)$ is decreasing and Schur-convex in u_i's. Taking the derivative of $\bar{F}_{Y_{2:n}}(t)$ with respect to u_k is

$$\begin{aligned}
\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial u_k} &= \frac{\partial g^{-1}(u_k)}{\partial u_k} \frac{1}{g^{-1}(u_k)} \left[\sum_{i \neq k}^{n} \prod_{j \neq i}^{n} g^{-1}(u_j) \bar{F}(t; \alpha_j, \beta_j) \right. \\
&\quad \left. -(n-1) \prod_{i=1}^{n} g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i) \right] \\
&= \frac{\partial \log g^{-1}(u_k)}{\partial u_k} \left[\sum_{i \neq k}^{n} \prod_{j \neq i}^{n} g^{-1}(u_j) \bar{F}(t; \alpha_j, \beta_j) - (n-1) \prod_{i=1}^{n} g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i) \right].
\end{aligned}$$

Since $g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i) \in [0, 1]$, for any $i = 1, 2, \ldots, n$, we have

$$\prod_{j \neq i}^{n} g^{-1}(u_j) \bar{F}(t; \alpha_j, \beta_j) \geq \prod_{i=1}^{n} g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i),$$

and then

$$\sum_{i \neq k}^{n} \prod_{j \neq i}^{n} g^{-1}(u_j) \bar{F}(t; \alpha_j, \beta_j) \geq \sum_{i \neq k}^{n} \prod_{i=1}^{n} g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i) = (n-1) \prod_{i=1}^{n} g^{-1}(u_i) \bar{F}(t; \alpha_i, \beta_i).$$

From condition (i), we have $\frac{\partial g^{-1}(u_k)}{\partial u_k} < 0$, which implies that $\bar{F}_{Y_{2:n}}(t)$ is decreasing in u_k's. We can compute

$$J(\boldsymbol{u}) := \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial u_1} - \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial u_2}$$

$$
\begin{aligned}
&= \frac{\partial \log g^{-1}(u_1)}{\partial u_1}\left[\sum_{i\neq 1}^{n}\prod_{j\neq i}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\alpha_i,\beta_i)\right] \\
&\quad -\frac{\partial \log g^{-1}(u_2)}{\partial u_2}\left[\sum_{i\neq 2}^{n}\prod_{j\neq i}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\alpha_i,\beta_i)\right] \\
&\geq \frac{\partial \log g^{-1}(u_2)}{\partial u_2}\left[\sum_{i\neq 1}^{n}\prod_{j\neq i}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j) - \sum_{i\neq 2}^{n}\prod_{j\neq i}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j)\right] \\
&= \frac{\partial \log g^{-1}(u_2)}{\partial u_2}\left[\prod_{j\neq 2}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j) - \prod_{j\neq 1}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j)\right] \\
&= \frac{\partial \log g^{-1}(u_2)}{\partial u_2}\prod_{j\neq 1, j\neq 2}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j)\Big[g^{-1}(u_1)\bar{F}(t;\alpha_1,\beta_1) \\
&\quad -g^{-1}(u_2)\bar{F}(t;\alpha_2,\beta_2)\Big] \\
&= \frac{\partial g^{-1}(u_2)}{\partial u_2}\frac{1}{g^{-1}(u_2)}\prod_{j\neq 1, j\neq 2}^{n} g^{-1}(u_j)\bar{F}(t;\alpha_j,\beta_j)\Big[g^{-1}(u_1)\bar{F}(t;\alpha_1,\beta_1) \\
&\quad -g^{-1}(u_2)\bar{F}(t;\alpha_2,\beta_2)\Big].
\end{aligned}
$$

The first inequality is true due to condition (ii). From condition (i), for $u_1 \geq u_2$, we have $g^{-1}(u_1) < g^{-1}(u_2)$. With conditions (iii) and (iv), for $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$, we also have

$$\bar{F}(t;\alpha_1,\beta_1) \leq \bar{F}(t;\alpha_2,\beta_1) \leq \bar{F}(t;\alpha_2,\beta_2).$$

By combining above observations, we readily observe that

$$
\begin{aligned}
& g^{-1}(u_1)\bar{F}(t;\alpha_1,\beta_1) < g^{-1}(u_2)\bar{F}(t;\alpha_2,\beta_2) \\
\Rightarrow\quad & g^{-1}(u_1)\bar{F}(t;\alpha_1,\beta_1) - g^{-1}(u_2)\bar{F}(t;\alpha_2,\beta_2) < 0.
\end{aligned}
$$

On the other hand, from assumption (i) and the fact that $\frac{\partial g^{-1}(u_2)}{\partial u_2} < 0$, it follows that

$$\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial u_1} - \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial u_2} \geq 0.$$

Thus, from Lemma 2.4, it follows that $\bar{F}_{Y_{2:n}}(t)$ is Schur-convex in u_i's. This completes the proof of the theorem. □

Remark 3.2. *It needs to be mentioned that the conditions "g is a differentiable and strictly decreasing function and g^{-1} is log-convex in $u \in \mathbb{R}_+$" in Theorem 3.1 are general and hold for many functions g. For example, consider some special cases in the Table 1 that satisfy in the conditions (i) and (ii) of Theorem 3.1.*

It should be mentioned that the Theorem 3.1 has a nice interpretation as follows: under some conditions, if $g(p_i)$'s are more dispersed than $g(p_i^*)$'s, then the survival function of fail-safe system with $g(p_i)$'s is larger than that with $g(p_i^*)$'s.

Table 1: Some special $g(p)$.

$g(p)$	Domain	Parameter Space
$(1-p)/p$	$(0,1)$	$-$
$-\log p$	$(0,1)$	$-$
$\theta^{-1}(p^{-\theta}-1)$	$(0,1)$	$\theta \in (0,\infty)$
$(1-\log p)^{\theta}-1$	$(0,1)$	$\theta \in (0,1]$

The following four corollaries can be obtained from Theorem 3.1 directly. The following mentioned semiparametric distributions in the following corollaries are very flexible family of distributions. For additional discussion about the these models, one may refer to Marshall and Olkin (2007). We develop results here by considering these as lifetime distributions of components.

Corollary 3.3. *Suppose the survival function of X is as follows*

$$\bar{F}(t;\alpha,\beta) = 1-\Big(F(\alpha t)\Big)^{\beta}, \qquad t>0, \alpha>0, \beta>0.$$

The partial derivatives of $\bar{F}(t;\alpha,\beta)$ with respect to α is

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \alpha} = -t\beta\Big(F(\alpha t)\Big)^{\beta}\tilde{r}(\alpha t) \le 0,$$

where $\tilde{r}$ is reversed hazard rate of baseline distribution, which means that $\bar{F}(t;\alpha,\beta)$ is decreasing in α and the condition (iii) of Theorem 3.1 is satisfied. We also have

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \beta} = -\Big(F(\alpha t)\Big)^{\beta}\log\Big(F(\alpha t)\Big) \ge 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is increasing in β and also condition (iv) of Theorem 3.1 is satisfied.

Corollary 3.4. *Consider the survival function of X as*

$$\bar{F}(t;\alpha,\beta) = 1-\Big(1-\big(\bar{F}(t)\big)^{\alpha}\Big)^{\beta}, \qquad t>0, \alpha>0, \beta>0.$$

The partial derivatives of $\bar{F}(t;\alpha,\beta)$ with respect to α is

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \alpha} = \beta\big(\bar{F}(t)\big)^{\alpha}\Big(1-\big(\bar{F}(t)\big)^{\alpha}\Big)^{\beta-1}\log\big(\bar{F}(t)\big) \le 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is decreasing in α and the condition (iii) of Theorem 3.1 is satisfied. We also have

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \beta} = -\Big(1-\big(\bar{F}(t)\big)^{\alpha}\Big)^{\beta}\log\Big(1-\big(\bar{F}(t)\big)^{\alpha}\Big) \ge 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is increasing in β and also condition (iv) of Theorem 3.1 is satisfied.

Corollary 3.5. *Suppose the survival function of X is given by*

$$\bar{F}(t;\alpha,\beta) = \Big(1 - F^{\beta}(t)\Big)^{\alpha}, \qquad t > 0, \alpha > 0, \beta > 0.$$

The partial derivatives of $\bar{F}(t;\alpha,\beta)$ with respect to α is

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \alpha} = \log\Big(1 - F^{\beta}(t)\Big)\Big(1 - F^{\beta}(t)\Big)^{\alpha} \leq 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is decreasing in α and the condition (iii) of Theorem 3.1 is satisfied. We also have

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \beta} = -\alpha\Big(F^{\beta}(t)\Big)\log\Big(F^{\beta}(t)\Big)\Big(1 - F^{\beta}(t)\Big)^{\alpha-1} \geq 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is increasing in β and also condition (iv) of Theorem 3.1 is satisfied.

Corollary 3.6. *Suppose the survival function of X is as follows*

$$\bar{F}(t;\alpha,\beta) = \frac{1 - \big(F(t)\big)^{\beta}}{1 - \bar{\alpha}\big(F(t)\big)^{\beta}}, \qquad t > 0, \alpha > 0, \beta > 0.$$

The partial derivatives of $\bar{F}(t;\alpha,\beta)$ with respect to α is

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \alpha} = -\frac{\big(F(t)\big)^{\beta}\Big(1 - \big(F(t)\big)^{\beta}\Big)}{\Big(1 - \bar{\alpha}\big(F(t)\big)^{\beta}\Big)^{2}} \leq 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is decreasing in α and the condition (iii) of Theorem 3.1 is satisfied. We also have

$$\frac{\partial \bar{F}(t;\alpha,\beta)}{\partial \beta} = -\frac{\alpha\big(F(t)\big)^{\beta}\log\big(F(t)\big)}{\Big(1 - \bar{\alpha}\big(F(t)\big)^{\beta}\Big)^{2}} \geq 0,$$

which means that $\bar{F}(t;\alpha,\beta)$ is increasing in β and also condition (iv) of Theorem 3.1 is satisfied.

The following numerical example provides an illustration of the result established in Theorem 3.1.

Example 3.7. *Let us consider the standard exponential as the baseline distribution in Corollary 3.4. Set $(\alpha_1,\alpha_2,\alpha_3) = (0.8, 0.4, 0.2)$, $(\beta_1,\beta_2,\beta_3) = (0.3, 1.1, 0.5)$, $(p_1,p_2,p_3) = (e^{-6}, e^{-1}, e^{-3})$ and $(p_1^*,p_2^*,p_3^*) = (e^{-5}, e^{-2}, e^{-4})$. With $g(p) = -\log p$, it is easy to observe that $\big(g(p_1), g(p_2), g(p_3)\big) \overset{w}{\succeq} \big(g(p_1^*), g(p_2^*), g(p_3^*)\big)$.*

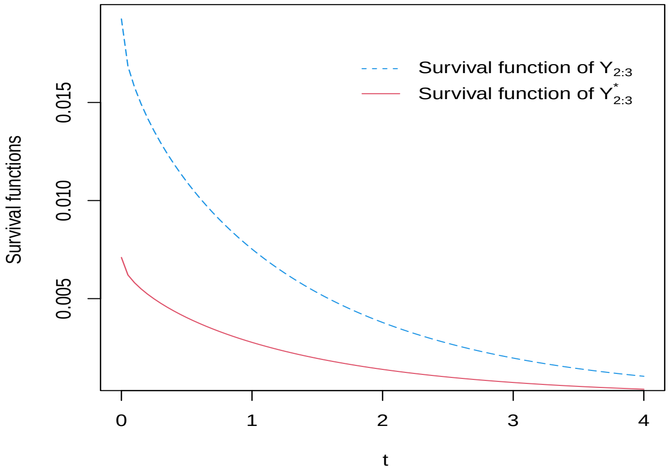

Figure 1: Plots of survival functions of $Y_{2:3}$ and $Y^*_{2:3}$.

We may question whether the result of Theorem 3.1 could hold if $(\boldsymbol{g(p)}, \boldsymbol{\beta}) \notin \mathcal{F}_n$? The following discussion provides a counterexample.

Let us consider the standard exponential as the baseline distribution in Corollary 3.5. Set $(\alpha_1, \alpha_2, \alpha_3) = (0.8, 0.4, 0.2)$, $(\beta_1, \beta_2, \beta_3) = (0.7, 0.5, 0.2)$, $(p_1, p_2, p_3) = (e^{-0.5}, e^{-0.4}, e^{-0.1})$ and $(p^*_1, p^*_2, p^*_3) = (e^{-0.7}, e^{-0.3}, e^{-0.2})$. With $g(p) = -\log p$, it is easy to observe that $\big(g(p_1), g(p_2), g(p_3)\big) \overset{w}{\succeq} \big(g(p^*_1), g(p^*_2), g(p^*_3)\big)$. But, the survival functions of $Y_{2:3}$ and $Y^*_{2:3}$ satisfy the following:

$$\begin{aligned}
\bar{F}_{Y_{2:3}}(0.002; \boldsymbol{g(p)}, \boldsymbol{\beta}) \approx 0.8702 \quad &> \quad 0.8686 \approx \bar{F}_{Y^*_{2:3}}(0.002; \boldsymbol{g(p^*)}, \boldsymbol{\beta}),\\
\bar{F}_{Y^*_{2:3}}(0.01; \boldsymbol{g(p^*)}, \boldsymbol{\beta}) \approx 0.8948 \quad &> \quad 0.8945 \approx \bar{F}_{Y_{2:3}}(0.01; \boldsymbol{g(p)}, \boldsymbol{\beta}).
\end{aligned}$$

This means that $Y_{2:3}$ and $Y^*_{2:3}$ cannot be compared in the usual stochastic order when $(\boldsymbol{g(p)}, \boldsymbol{\beta}) \notin \mathcal{F}_n$, because it can be seen that the survival function of $Y_{2:3}$ and $Y^*_{2:3}$ cross each other.

Next theorem shows that under some conditions, if β_i's are more dispersed than β^*_i's, then the survival function of fail-safe system with β_i's is larger than that with β^*_i's. Note that $g : [0,1] \longmapsto \mathbb{R}_+$ is considered as a differentiable and strictly increasing function in the following theorem. Let us set

$$\mathcal{F}_n = \left\{ (\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{pmatrix} : a_i, b_j > 0 \ \textit{and} \ (a_i - a_j)(b_i - b_j) \le 0, \ i, j = 1, \dots, n \right\}.$$

Theorem 3.8. *Suppose $X_1, \dots, X_n (X^*_1, \dots, X^*_n)$ are independent nonnegative random variables with $X_i \sim \bar{F}(\cdot; \beta_i)(X^*_i \sim \bar{F}(\cdot; \beta^*_i))$, where $\bar{F}(\cdot; \beta_i)(\bar{F}(\cdot; \beta^*_i))$ denotes the survival function of $X_i(X^*_i)$, and $\beta_i > 0(\beta^*_i > 0)$ is the distribution parameter of $X_i(X^*_i)$ for $i = 1, 2, \dots, n$. Let $I_{p_1}, \dots, I_{p_n}$ be a set of independent Bernoulli random variables, independent of X_i's and X^*_i's, with $\mathbb{E}[I_{p_i}] = p_i, i = 1, 2, \dots, n$. Let $Y_i = I_{p_i} X_i$ and $Y^*_i = I_{p_i} X^*_i, i = 1, 2, \dots, n$. Assume that the following conditions hold:*
(i) $g : [0,1] \longmapsto \mathbb{R}_+$ is a differentiable and strictly increasing function;

(ii) $\bar{F}(\cdot;\beta)$ *is decreasing in* $\beta \in \mathbb{R}_+$*;*
(iii) $\bar{F}(\cdot;\beta)$ *is log-convex in* $\beta \in \mathbb{R}_+$*;*
Then, for $(\boldsymbol{g(p)}, \boldsymbol{\beta}) \in \mathcal{F}_n$ *and* $(\boldsymbol{g(p)}, \boldsymbol{\beta^*}) \in \mathcal{F}_n$*, we have*

$$\left(\beta_1, \beta_2, \ldots, \beta_n\right) \overset{w}{\succeq} \left(\beta_1^*, \beta_2^*, \ldots, \beta_n^*\right) \Longrightarrow Y_{2:n} \geq_{st} Y_{2:n}^*.$$

Proof. The assumption that $(\boldsymbol{g(p)}, \boldsymbol{\beta}) \in \mathcal{F}_n$ implies that $(g(p_i) - g(p_j))(\beta_i - \beta_j) \leq 0$, which means that $g(p_1) \geq g(p_2) \geq \ldots \geq g(p_n)$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, or $g(p_1) \leq g(p_2) \leq \ldots \leq g(p_n)$ and $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$. Here, we present the proof only for the case when $g(p_1) \geq g(p_2) \geq \ldots \geq g(p_n)$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, since the proof for the other case is quite similar. Let us set $g(p_i) = u_i$ for $i = 1, 2, \ldots, n$ and g^{-1} denotes the inverse function of g. Based on Theorem 3.1, we have

$$\bar{F}_{Y_{2:n}}(t) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\beta_i), \qquad t \geq 0.$$

To obtain the desired result, according to Lemma 2.5, it suffices to show that for each fixed $t > 0$, $\bar{F}_{Y_{2:n}}(t)$ is decreasing and Schur-convex in β_i's. The partial derivative of $\bar{F}_{Y_{2:n}}(t)$ with respect to β_k is given by

$$\begin{aligned}\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_k} &= \frac{\partial \bar{F}(t;\beta_k)}{\partial \beta_k}\frac{1}{\bar{F}(t;\beta_k)}\left[\sum_{i \neq k}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\beta_i)\right] \\ &= \frac{\partial \log \bar{F}(t;\beta_k)}{\partial \beta_k}\left[\sum_{i \neq k}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\beta_i)\right].\end{aligned}$$

From Theorem 3.1, we know that

$$\sum_{i \neq k}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\beta_i) \geq 0.$$

Thus, conditions (ii) implies that $\bar{F}_{Y_{2:n}}(t)$ is decreasing in β_i's. Now, in a similar method, we can compute that

$$\begin{aligned}I(\boldsymbol{\beta}) :=& \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_1} - \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_2} \\ =& \frac{\partial \log \bar{F}(t;\beta_1)}{\partial \beta_1}\left[\sum_{i \neq 1}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\beta_i)\right] \\ &- \frac{\partial \log \bar{F}(t;\beta_2)}{\partial \beta_2}\left[\sum_{i \neq 2}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - (n-1)\prod_{i=1}^{n} g^{-1}(u_i)\bar{F}(t;\beta_i)\right] \\ \leq& \frac{\partial \log \bar{F}(t;\beta_2)}{\partial \beta_2}\left[\sum_{i \neq 1}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - \sum_{i \neq 2}^{n}\prod_{j \neq i}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j)\right] \\ =& \frac{\partial \log \bar{F}(t;\beta_2)}{\partial \beta_2}\left[\prod_{j \neq 2}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j) - \prod_{j \neq 1}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j)\right]\end{aligned}$$

$$= \frac{\partial \log \bar{F}(t;\beta_2)}{\partial \beta_2} \prod_{j\neq 1, j\neq 2}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j)\Big[g^{-1}(u_1)\bar{F}(t;\beta_1) - g^{-1}(u_2)\bar{F}(t;\beta_2)\Big]$$

$$= \frac{\partial \bar{F}(t;\beta_2)}{\partial \beta_2}\frac{1}{\bar{F}(t;\beta_2)} \prod_{j\neq 1, j\neq 2}^{n} g^{-1}(u_j)\bar{F}(t;\beta_j)\Big[g^{-1}(u_1)\bar{F}(t;\beta_1) - g^{-1}(u_2)\bar{F}(t;\beta_2)\Big],$$

where first inequality is due to conditions (iii), for $\beta_1 \leq \beta_2$, and the the following fact

$$\frac{\partial \log \bar{F}(t,\beta_1)}{\partial \beta_1} \leq \frac{\partial \log \bar{F}(t,\beta_2)}{\partial \beta_2} \leq 0.$$

Further, for $u_1 \geq u_2$ and $\beta_1 \leq \beta_2$, it follows that

$$g^{-1}(u_1)\bar{F}(t;\beta_1) - g^{-1}(u_2)\bar{F}(t;\beta_2) > 0.$$

Then, we readily observe that

$$\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_1} - \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_2} \leq 0,$$

or by the assumption that $\beta_1 \leq \beta_2$, we have

$$(\beta_1 - \beta_2)\Big\{\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_1} - \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_2}\Big\} \geq 0.$$

Thus, from Lemma 2.4, it follows that $\bar{F}_{Y_{2:n}}(t)$ is Schur-convex in β_i's. This completes the proof of the theorem. □

Remark 3.9. *It should be mentioned that the condition "g is a differentiable and strictly increasing function" in Theorem 3.8 is general and holds for many functions g. For example, for the cases when $g(p) = p/(1-p)$, $g(p) = \log(1+p)$ and also $g(p) = p^\theta$ for $\theta > 0$, we can readily show that h is a strictly increasing function.*

The following corollaries can be derived from Theorem 3.8 for the cases of proportional hazard rate and scale models.

Corollary 3.10. *Suppose X follows proportional hazard rate model with survival function*

$$\bar{F}(t;\beta) = \bar{F}^{\beta}(t), \qquad t > 0, \beta > 0.$$

The partial derivatives of $\bar{F}(t;\beta)$ with respect to β is

$$\frac{\partial \bar{F}(t;\beta)}{\partial \beta} = \log(\bar{F}(t))\bar{F}^{\beta}(t) \leq 0,$$

which means that $\bar{F}(t;\beta)$ is decreasing in β and the condition (ii) of Theorem 3.8 is satisfied. We also have

$$\frac{\partial^2 \log \bar{F}(t;\beta)}{\partial \beta^2} = 0,$$

which means that $\bar{F}(t;\beta)$ is log-convex in β and also condition (iii) of Theorem 3.8 is satisfied.

Corollary 3.11. *Consider the survival function of scale model with*

$$\bar{F}(t;\beta) = 1 - F(\beta t), \qquad t > 0, \beta > 0.$$

The partial derivatives of $\bar{F}(t;\beta)$ with respect to β is

$$\frac{\partial \bar{F}(t;\beta)}{\partial \beta} = -xf(\beta t) \leq 0,$$

which means that $\bar{F}(t;\beta)$ is decreasing in α and the condition (ii) of Theorem 3.8 is satisfied. We also have

$$\frac{\partial^2 \log \bar{F}(t;\beta)}{\partial \beta^2} = -t^2 r'(\beta t),$$

where $r(t)$ is hazard rate function of baseline distribution, which means that $\bar{F}(t;\beta)$ is log-convex in β if $r(t)$ is decreasing. Thus, the condition (iii) of Theorem 3.8 is also satisfied.

The following numerical example provides an illustration of the result established in Theorem 3.8.

Example 3.12. *Let Weibull be the baseline survival function with $\bar{F}(t) = e^{-t^2}$, for $t > 0$, in Corollary 3.10. Further, Set $(\beta_1, \beta_2, \beta_3) = (0.1, 1, 9)$, $(\beta_1^*, \beta_2^*, \beta_3^*) = (0.2, 4, 6)$ and $(p_1, p_2, p_3) = (e^{0.6} - 1, e^{0.3} - 1, e^{-0.4} - 1)$. With $g(p) = \log(1 + p)$, it is easy to observe that $\left(\beta_1, \beta_2, \beta_3\right) \overset{w}{\succeq} \left(\beta_1^*, \beta_2^*, \beta_3^*\right)$.*

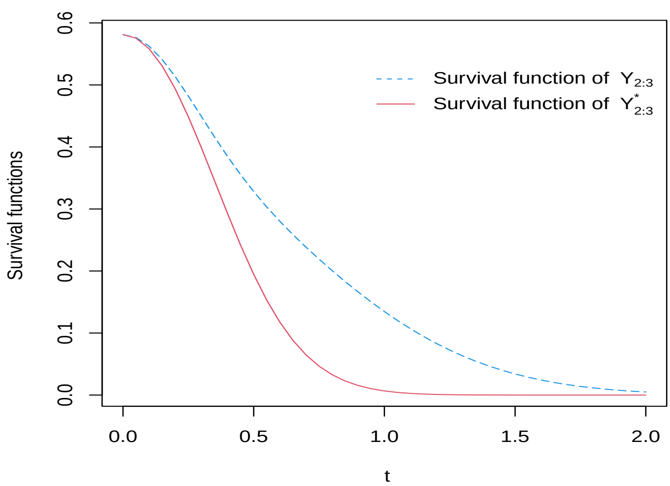

Figure 2: Plots of survival functions of $Y_{2:3}$ and $Y_{2:3}^*$.

The survival functions of $Y_{2:3}$ and $Y_{2:3}^*$ are plotted in Figure 2, which confirms that $Y_{2:3} \geq_{st} Y_{2:3}^*$.

4 Stochastic comparisons with dependent components

Most of the existing literature on stochastic comparisons of fail-safe systems have focused on the case when the components in the systems are all independent. However, when such technical systems are in operation, many important factors, such as operating conditions, environmental conditions and stress factors, are shared and experienced by all the components in the system. It would, therefore, be reasonable for the lifetimes of components in a system to be dependent. Now, let us set

$$\mathcal{F}_n = \left\{(\boldsymbol{a}, \boldsymbol{b}) = \left(\begin{smallmatrix} a_1 \dots a_n \\ b_1 \dots b_n \end{smallmatrix} \right) : a_i, b_j > 0 \ \ and \ \ (a_i - a_j)(b_i - b_j) \le 0, \ i, j = 1, \dots, n \right\}.$$

Theorem 4.1. *Suppose $X_1, \dots, X_n (X_1^*, \dots, X_n^*)$ are dependent nonnegative random variables with $X_i \sim \bar{F}(\cdot; \beta_i)(X_i^* \sim \bar{F}(\cdot; \beta_i^*))$ with Archimedean copula with generator ϕ, where $\bar{F}(\cdot; \beta_i)(\bar{F}(\cdot; \beta_i^*))$ denotes the survival function of $X_i(X_i^*)$, and $\beta_i > 0(\beta_i^* > 0)$ is the distribution parameter of $X_i(X_i^*)$ for $i = 1, 2, \dots, n$. Let $I_{p_1}, \dots, I_{p_n}$ be a set of independent Bernoulli random variables, independent of X_i's and X_i^*'s, with $\mathbb{E}[I_{p_i}] = p_i, i = 1, 2, \dots, n$. Let $Y_i = I_{p_i} X_i$ and $Y_i^* = I_{p_i} X_i^*, i = 1, 2, \dots, n$. Assume that the following conditions hold:*
(i) $g : [0, 1] \longmapsto \mathbb{R}_+$ is a differentiable and strictly increasing function;
(ii) $\bar{F}(\cdot; \beta)$ is decreasing and log-convex in $\beta \in \mathbb{R}_+$;
(iii) $\psi(t)$ is log-concave in t;
Then, for $(\boldsymbol{g(p)}, \boldsymbol{\beta}) \in \mathcal{F}_n$ and $(\boldsymbol{g(p)}, \boldsymbol{\beta^*}) \in \mathcal{F}_n$, we have

$$(\beta_1, \dots, \beta_n) \overset{w}{\succeq} (\beta_1^*, \dots, \beta_n^*) \Longrightarrow Y_{2:n} \ge_{st} Y_{2:n}^*.$$

Proof. The assumption that $(\boldsymbol{g(p)}, \boldsymbol{\beta}) \in \mathcal{F}_n$ implies that $(g(p_i) - g(p_j))(\beta_i - \beta_j) \le 0$, which means that $g(p_1) \ge g(p_2) \ge \dots \ge g(p_n)$ and $\beta_1 \le \beta_2 \le \dots \le \beta_n$, or $g(p_1) \le g(p_2) \le \dots \le g(p_n)$ and $\beta_1 \ge \beta_2 \ge \dots \ge \beta_n$. Here, we present the proof only for the case when $g(p_1) \ge g(p_2) \ge \dots \ge g(p_n)$ and $\beta_1 \le \beta_2 \le \dots \le \beta_n$, since the proof for the other case is quite similar. Let us set $g(p_i) = u_i$ for $i = 1, 2, \dots, n$ and g^{-1} denotes the inverse function of g. The survival function of $Y_{2:n}$, for $t \ge 0$, can be written as

$$\begin{aligned}
\bar{F}_{Y_{2:n}}(t) &= \mathbb{P}(Y_{2:n} > t) \\
&= \sum_{i=1}^{n} \mathbb{P}(Y_i \le t, Y_j > t, for j \ne i) + \mathbb{P}(Y_1 > t, \dots, Y_n > t) \\
&= \sum_{i=1}^{n} \Big[\mathbb{P}(Y_j > t, for j \ne i) - \mathbb{P}(Y_j > t, for j = 1, \dots, n) \Big] \\
&\quad + P(Y_1 > t, \dots, Y_n > t) \\
&= \sum_{i=1}^{n} \mathbb{P}(Y_j > t, for j \ne i) - (n-1) P(Y_1 > t, \dots, Y_n > t) \\
&= \sum_{i=1}^{n} (\prod_{j \ne i}^{n} p_j) \mathbb{P}(X_j > t, for j \ne i) - (n-1)(\prod_{i=1}^{n} p_i) P(X_1 > t, \dots, X_n > t) \\
&= \sum_{i=1}^{n} \left(\prod_{j \ne i}^{n} g^{-1}(u_j) \right) \left(\psi \left[\sum_{j \ne i}^{n} \phi(\bar{F}(t; \beta_i)) \right] \right)
\end{aligned}$$

$$-(n-1)\left(\prod_{i=1}^{n} g^{-1}(u_i)\right)\left(\psi\left[\sum_{i=1}^{n}\phi(\bar{F}(t;\beta_i))\right]\right).$$

To obtain the desired result, according to Lemma 2.5, it suffices to show that for each fixed $t \geq 0$, $\bar{F}_{Y_{2:n}}(t)$ is decreasing and Schur-convex in β_i's. Taking the derivative of $\bar{F}_{Y_{2:n}}(t)$ with respect to β_k is given by

$$\begin{aligned}\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_k} &= \left[\sum_{i\neq k}^{n}\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right)\right.\\ &\quad\left. -(n-1)\left(\prod_{i=1}^{n} g^{-1}(u_i)\right)\left(\psi'\left[\sum_{i=1}^{n}\phi(\bar{F}(t;\beta_i))\right]\right)\right]\\ &\quad \times\frac{\psi(\phi(\bar{F}(t;\beta_k)))}{\psi'(\phi(\bar{F}(t;\beta_k)))}\times\frac{\partial\log(\bar{F}(t;\beta_k))}{\partial\beta_k}.\end{aligned}$$

It is easy to observe that

$$\begin{aligned}\prod_{j\neq i}^{n} g^{-1}(u_j) &\geq \prod_{m=1}^{n} g^{-1}(u_m)\geq 0, &&\text{for } i\in\{1,2,\ldots,n\} \quad (1)\\ \sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j)) &\leq \sum_{m=1}^{n}\phi(\bar{F}(t;\beta_m)), &&\text{for } i\in\{1,2,\ldots,n\}\end{aligned}$$

and then

$$\psi'(\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j)))\leq\psi'(\sum_{m=1}^{n}\phi(\bar{F}(t;\beta_m)))\leq 0, \qquad \text{(Because } \psi''>0). \quad (2)$$

So, according to (1) and (2), we can observe that

$$\begin{aligned}A &= \sum_{i\neq k}^{n}\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right)\\ &\quad -(n-1)\left(\prod_{i=1}^{n} g^{-1}(u_i)\right)\left(\psi'\left[\sum_{i=1}^{n}\phi(\bar{F}(t;\beta_i))\right]\right)\\ &= \sum_{i\neq k}^{n}\left\{\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right)\right.\\ &\quad\left. -\left(\prod_{i=1}^{n} g^{-1}(u_i)\right)\left(\psi'\left[\sum_{i=1}^{n}\phi(\bar{F}(t;\beta_i))\right]\right)\right\}\leq 0.\end{aligned}$$

Since $\bar{F}(x;\beta_i)$ is decreasing with respect to β_i, it follows that

$$\frac{\partial\log(\bar{F}(t;\beta_i))}{\partial\beta_i}\leq 0.$$

By combining the above results, we can conclude that $\bar{F}_{Y_{2:n}}(t)$ is decreasing in β_i's. Now, we have

$$
\begin{aligned}
J(\boldsymbol{\beta}) &:= \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_1} - \frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_2} \\
&= \Bigg[\sum_{i\neq 1}^{n}\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right) \\
&\quad -(n-1)\left(\prod_{i=1}^{n} g^{-1}(u_i)\right)\left(\psi'\left[\sum_{i=1}^{n}\phi(\bar{F}(t;\beta_i))\right]\right)\Bigg] \\
&\quad \times \frac{\psi(\phi(\bar{F}(t;\beta_1)))}{\psi'(\phi(\bar{F}(t;\beta_1)))}\frac{\partial \log(\bar{F}(t;\beta_1))}{\partial \beta_1} \\
&\quad -\Bigg[\sum_{i\neq 2}^{n}\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right) \\
&\quad -(n-1)\left(\prod_{i=1}^{n} g^{-1}(u_i)\right)\left(\psi'\left[\sum_{i=1}^{n}\phi(\bar{F}(t;\beta_i))\right]\right)\Bigg] \\
&\quad \times \frac{\psi(\phi(\bar{F}(t;\beta_2)))}{\psi'(\phi(\bar{F}(t;\beta_2)))}\frac{\partial \log(\bar{F}(t;\beta_2))}{\partial \beta_2} \\
&\leq \Bigg\{\sum_{i\neq 1}^{n}\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right) \\
&\quad -\sum_{i\neq 2}^{n}\left(\prod_{j\neq i}^{n} g^{-1}(u_j)\right)\left(\psi'\left[\sum_{j\neq i}^{n}\phi(\bar{F}(t;\beta_j))\right]\right)\Bigg\} \\
&\quad \times \frac{\psi(\phi(\bar{F}(t;\beta_1)))}{\psi'(\phi(\bar{F}(t;\beta_1)))}\frac{\partial \log(\bar{F}(t;\beta_1))}{\partial \beta_1} \\
&= \left\{g^{-1}(u_1)\left(\psi'\left[\sum_{j\neq 2}^{n}\phi(\bar{F}(t;\beta_j))\right]\right) - g^{-1}(u_2)\left(\psi'\left[\sum_{j\neq 1}^{n}\phi(\bar{F}(t;\beta_j))\right]\right)\right\} \\
&\quad \times \left(\prod_{j\neq\{1,2\}}^{n} g^{-1}(u_j)\right)\frac{\psi(\phi(\bar{F}(t;\beta_1)))}{\psi'(\phi(\bar{F}(t;\beta_1)))} \times \frac{\partial \log(\bar{F}(t;\beta_1))}{\partial \beta_1} \\
&\overset{sgn}{=} \left\{g^{-1}(u_1)\left(\psi'\left[\sum_{j\neq 2}^{n}\phi(\bar{F}(t;\beta_j))\right]\right) - g^{-1}(u_2)\left(\psi'\left[\sum_{j\neq 1}^{n}\phi(\bar{F}(t;\beta_j))\right]\right)\right\} \\
&\quad \times \frac{\psi(\phi(\bar{F}(t;\beta_1)))}{\psi'(\phi(\bar{F}(t;\beta_1)))}\frac{\partial \log(\bar{F}(t;\beta_1))}{\partial \beta_1} \leq 0.
\end{aligned}
$$

The first and second inequalities are true due to following reasons: Since $\bar{F}(x;\beta)$ is decreasing in β and ϕ is also decreasing, for $\beta_1 \leq \beta_2$, we have

$$\phi(\bar{F}(t;\beta_1)) \leq \phi(\bar{F}(t;\beta_2)). \tag{3}$$

Thus, it holds that

$$\begin{aligned}\sum_{j\neq 1}^{n}\phi(\bar{F}(t;\beta_j)) &= \sum_{j=1}^{n}\phi(\bar{F}(t;\beta_j))-\phi(\bar{F}(t;\beta_1))\\ &\geq \sum_{j=1}^{n}\phi(\bar{F}(t;\beta_j))-\phi(\bar{F}(t;\beta_2))\\ &= \sum_{j\neq 2}^{n}\phi(\bar{F}(t;\beta_j)).\end{aligned}$$

From convexity of ψ, we immediately conclude that $0 \geq \psi'\left[\sum_{j\neq 1}^{n}\phi(\bar{F}(t;\beta_j))\right] \geq \psi'\left[\sum_{j\neq 2}^{n}\phi(\bar{F}(t;\beta_j))\right]$ and then

$$0 \leq -\psi'\left[\sum_{j\neq 1}^{n}\phi(\bar{F}(t;\beta_j))\right] \leq -\psi'\left[\sum_{j\neq 2}^{n}\phi(\bar{F}(t;\beta_j))\right]. \tag{4}$$

On the other hand, since g is increasing, g^{-1} is also increasing and then for $u_1 \geq u_2$, we have

$$u_1 \geq u_2 \Longrightarrow 0 \leq g^{-1}(u_2) \leq g^{-1}(u_1). \tag{5}$$

By combining (4) and (5), we have

$$g^{-1}(u_2)\psi'\left[\sum_{j\neq 1}^{n}\phi(\bar{F}(t;\beta_j))\right] \geq g^{-1}(u_1)\psi'\left[\sum_{j\neq 2}^{n}\phi(\bar{F}(t;\beta_j))\right]. \tag{6}$$

Because ψ is log-concave (or equivalently, ψ/ψ' is increasing), based on (3), it follows that

$$\frac{\psi(\phi(\bar{F}(t;\beta_1)))}{\psi'(\phi(\bar{F}(t;\beta_1)))} \leq \frac{\psi(\phi(\bar{F}(t;\beta_2)))}{\psi'(\phi(\bar{F}(t;\beta_2)))} \leq 0. \tag{7}$$

Since $\bar{F}(x;\beta)$ is decreasing and log-convex with respect to β, by the assumption that $\beta_1 \leq \beta_2$, we also have

$$\frac{\partial \log(\bar{F}(t;\beta_1))}{\partial \beta_1} \leq \frac{\partial \log(\bar{F}(t;\beta_2))}{\partial \beta_2} \leq 0. \tag{8}$$

From (7) and (8), we then get

$$\frac{\psi(\phi(\bar{F}(t;\beta_1)))}{\psi'(\phi(\bar{F}(t;\beta_1)))}\frac{\partial \log(\bar{F}(t;\beta_1))}{\partial \beta_1} \geq \frac{\psi(\phi(\bar{F}(t;\beta_2)))}{\psi'(\phi(\bar{F}(t;\beta_2)))}\frac{\partial \log(\bar{F}(t;\beta_2))}{\partial \beta_2} \geq 0. \tag{9}$$

Now, From (6) and (9), we conclude that $J(\boldsymbol{\beta}) \leq 0$ and then

$$(\beta_1-\beta_2)\left(\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_1}-\frac{\partial \bar{F}_{Y_{2:n}}(t)}{\partial \beta_2}\right) \geq 0,$$

which, from Lemma 2.4, it follows that $\bar{F}_{Y_{2:n}}(t)$ is Schur-convex in β_i's. This completes the proof of the theorem. □

Remark 4.2. *It should be mentioned that the condition "ψ is log-concave" in Theorem 4.1 is quite general and is easy to verify for many well-known Archimedean copulas. For example, consider*
(i) the Hougaard copula with the generator $\psi(t) = e^{1-(1+t)^{\theta}}$ for $\theta \geq 1$. Note that $\log \psi(t) = 1 - (1+t)^{\theta}$ and then $[\log \psi(t)]'' = -\theta(\theta-1)(1+t)^{\theta-2}$ is nonpositve in $t \in [0,1]$, for $\theta \geq 1$, which means that ψ is log-concave in $t \in [0,1]$;
Next, let us consider
(ii) the Gumbel-Hougaard copula with generator $\psi(t) = e^{\frac{1}{\theta}(1-e^{t})}$, for $\theta \in (0, 0.5(3-\sqrt{5}))$. It can be observed that $\log \psi(t) = \frac{1}{\theta}(1-e^{t})$ and then $[\log \psi(t)]'' = -\frac{1}{\theta}e^{t}$ is nonpositive in $t \in [0,1]$, for $\theta \in (0, 0.5(3-\sqrt{5}))$, which means that ψ is log-concave in $t \in [0,1]$.

The following numerical example provides an illustration of the result established in Theorem 4.1.

Example 4.3. *Let us consider the standard exponential as the baseline distribution in Theorem 3.4. Set $(\beta_1, \beta_2, \beta_3) = (0.2, 0.4, 0.8)$, $(\beta_1^*, \beta_2^*, \beta_3^*) = (0.3, 0.3, 1.1)$ and $(p_1, p_2, p_3) = (e^{0.6}-1, e^{0.3}-1, e^{0.1}-1)$. With $g(p) = \log(1+p)$, it is easy to observe that $\left(\beta_1, \beta_2, \beta_3\right) \overset{w}{\succeq} \left(\beta_1^*, \beta_1^*, \beta_1^*\right)$. Now, we choose the Hougaard copula with the generator $\psi(t) = e^{1-(1+t)^{\theta}}$ with $\theta \geq 1$. Then, the survival function of $Y_{2:3}$ and $Y_{2:3}^*$, for $t \geq 0$ respectively, are given by*

$$
\begin{aligned}
\bar{F}_{Y_{2:3}}(t) &= g^{-1}(u_2)g^{-1}(u_3)\exp\left\{\left[(1+\beta_2 x)^{1/\theta} + (1+\beta_3 x)^{1/\theta} - 1\right]^{\theta}\right\} \\
&\quad + g^{-1}(u_1)g^{-1}(u_3)\exp\left\{\left[(1+\beta_1 x)^{1/\theta} + (1+\beta_3 x)^{1/\theta} - 1\right]^{\theta}\right\} \\
&\quad + g^{-1}(u_1)g^{-1}(u_2)\exp\left\{\left[(1+\beta_1 x)^{1/\theta} + (1+\beta_2 x)^{1/\theta} - 1\right]^{\theta}\right\} \\
&\quad - 2g^{-1}(u_1)g^{-1}(u_2)g^{-1}(u_3)\exp\left\{\left[(1+\beta_1 x)^{1/\theta} + (1+\beta_2 x)^{1/\theta}\right.\right. \\
&\quad \left.\left. + (1+\beta_3 x)^{1/\theta} - 2\right]^{\theta}\right\} \\
\bar{F}_{Y_{2:3}^*}(t) &= g^{-1}(u_2)g^{-1}(u_3)\exp\left\{\left[(1+\beta_2^* x)^{1/\theta} + (1+\beta_3^* x)^{1/\theta} - 1\right]^{\theta}\right\} \\
&\quad + g^{-1}(u_1)g^{-1}(u_3)\exp\left\{\left[(1+\beta_1^* x)^{1/\theta} + (1+\beta_3^* x)^{1/\theta} - 1\right]^{\theta}\right\} \\
&\quad + g^{-1}(u_1)g^{-1}(u_2)\exp\left\{\left[(1+\beta_1^* x)^{1/\theta} + (1+\beta_2^* x)^{1/\theta} - 1\right]^{\theta}\right\} \\
&\quad - 2g^{-1}(u_1)g^{-1}(u_2)g^{-1}(u_3)\exp\left\{\left[(1+\beta_1^* x)^{1/\theta} + (1+\beta_2^* x)^{1/\theta}\right.\right. \\
&\quad \left.\left. + (1+\beta_3^* x)^{1/\theta} - 2\right]^{\theta}\right\}.
\end{aligned}
$$

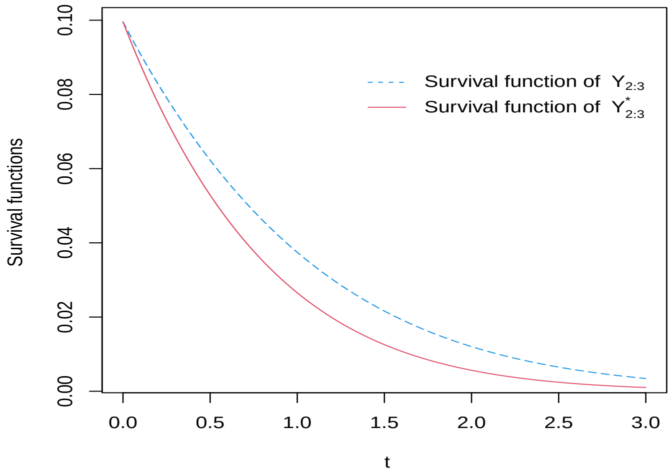

Figure 3: Plots of survival functions of $Y_{2:3}$ and $Y^*_{2:3}$ with Hougaard copula.

The survival functions of $Y_{2:3}$ and $Y^*_{2:3}$, for $\theta = 2$, are plotted in Figure 3, which implies that $Y_{2:3} \geq_{st} Y^*_{2:3}$.

5 Concluding remarks

A fail-safe is a special a design feature that, when a failure occurs, will respond in a way that no harm happens to the system itself. An example of a fail-safe system is an elevator in which brakes are held off brake pads by tension and if the tension gets lost, the brakes latch on the rails in the shaft thus preventing the elevator from falling. There are many similar fail-safe systems in day to day use.

In this paper, under some conditions and by using the concept of vector majorization and related orders, we have discussed stochastic comparisons of fail-safe systems under random shocks in the sense of usual stochastic order.

Here, we have obtained results for the usual stochastic order. Hazard rate, reversed hazard rate and likelihood ratio orders of these results can be considered as a topic for future research.

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