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E-Bayesian and hierarchical Bayesian estimation of reliability in multicomponent stress-strength model based on inverse Rayleigh distribution

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Abstract: This study focuses on estimating the reliability of a multicomponent stress-strength model using two Bayesian approaches: E-Bayesian and hierarchical Bayesian. This model follows Inverse Rayleigh distributions with distinct parameters. Additionally, the efficiency of the proposed methods is compared by employing Monte Carlo simulation and analyzing a data set.

Keywords: E-Bayesian estimation; Hierarchical Bayesian estimation; Inverse Rayleigh distribution; Multicomponent stress-strength model; Reliability.
Mathematics Subject Classification (2010): 62F15, 62N05.

1 Introduction

The inverse Rayleigh (IR) distribution has found applications in various fields of science and technology, including acoustics (Khan and King, 2015). Several authors, such as Gharraph (1993), Mukarjee and Maitim (1996), Guobing (2015), Abdel-Monem (2003), and Soliman et al. (2010), have proposed different methods for estimating the distribution parameters of the IR distribution. In this study, we present the probability density function and cumulative distribution function (cdf) of the IR distribution ($IR(\theta)$), respectively, as follows

$$f(x;\theta) = \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}}, \qquad x > 0,\ \theta > 0, \tag{1}$$

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$$F(x;\theta) = e^{-\frac{\theta}{x^2}}, \qquad x>0,\ \theta>0, \tag{2}$$

in which θ is the scale parameter. Consider a multicomponent system with k components, where the strengths of each component are represented by independently and identically distributed random variables $X_1,\ldots,X_k$. Each component is subjected to a random stress variable Y. The system is considered operational only if at least s (where $s<k$) strengths exceed the stress.

Let $Y, X_1,\ldots,X_k$ be independent random samples, where $G(y)$ represents the continuous distribution function of Y, and $F(x)$ represents the common continuous distribution function of $X_1,\ldots,X_k$. The reliability of a multicomponent stress-strength model, developed by Bhattacharyya and Johnson (1974), is given by

$$\begin{aligned} R_{s,k} &= P\,(\text{at Least s of the }\ (X_1,\ldots,X_k)\,\text{exceed } Y) \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}(1-F(y))^i\,(F(y))^{k-i}\,dG(y), \end{aligned} \tag{3}$$

where $X_1,\ldots,X_k$ are independent random variables with a cdf $F(x)$ and are subject to the common random stress Y. The reliability in a multicomponent stress-strength model, as defined in (1), has been studied by Bhattacharyya and Johnson (1974). Various authors have also investigated the reliability of single component stress-strength models for different distributions. These authors include Enis and Geisser (1971), Downtown (1973), Awad and Gharraf (1986), McCool (1991), Nandi and Aich (1996), Surles and Padgett (1998), Raqab and Kundu (2005), Kundu and Gupta (2005), Kundu and Gupta (2006), Raqab and Kundu (2005), Kundu and Raqab (2009), Asgharzadeh et al. (2011), Lio and Tsai (2012), Al-Mutairi et al. (2013), and Ghitany et al. (2015). In recent years, several authors have considered the estimation of reliability in multicomponent stress-strength systems as proposed by Bhattacharyya and Johnson (1974) and Pandey and Uddin (1985). For instance, Rao and Kantam (2010), Rao (2012), Rao (2012), and Rao et al. (2017) have utilized these methods to estimate the reliability of multicomponent stress-strength systems for different distributions such as log-logistic, generalized exponential, and Rayleigh.

The hierarchical Bayesian prior distribution was initially introduced by Lindley and Smith (1972) and later examined by Han (1997). Subsequently, E-Bayesian and hierarchical Bayesian methods were introduced. Recently, Han (2006) and Han (2011) employed E-Bayesian and hierarchical Bayesian methods to estimate the exponential distribution parameter and the reliability of the binomial distribution, respectively. Jaheen and Okasha (2011) utilized these methods to estimate the reliability of the Type 12 distribution based on Type II progressive censoring samples. Yousefzadeh (2017) employed them to estimate the Pascal distribution parameter, while Yaghoobzadeh (2019) utilized them to estimate the scale parameter of the Gompertz distribution under type II censoring schemes based on fuzzy data. Yaghoobzade and Makhdoom (2021) obtained E-Bayesian and hierarchical Bayesian estimations for $R=P(X{>}Y)$ in the Weibull distribution.

The remainder of the paper is structured as follows: This study focuses on obtaining E-Bayesian estimation and hierarchical Bayesian estimation of $R_{s,k}$ using the square error loss (SEL) function $L\left(\hat{\theta},\theta\right)=(\hat{\theta}-\theta)^2$ in Section 2. In Section 3, a simulation study is conducted using the Monte Carlo method. Additionally, we demonstrate the estimation process using two real data sets in Section 4. Finally, we conclude the paper in Section 5.

2 Methods of estimating $R_{s,k}$

Let us consider independent random variables X (stress) and Y (strength) that follow the IR distributions with parameters θ_1 and θ_2 respectively. Using equation (3), we can determine the reliability of a multicomponent stress-strength system for the IR distribution as follows.

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty \left(1 - e^{-\frac{\theta_1}{y^2}}\right)^i \left(e^{-\frac{\theta_1}{y^2}}\right)^{k-i} \left(\frac{2\theta_2}{y^3} e^{-\frac{\theta_2}{y^2}}\right) dy.$$

By assuming $v = \frac{\theta_2}{\theta_1}$ and introducing a variable transformation $t = e^{-\frac{\theta_1}{y^2}}$, we have

$$\begin{aligned} R_{s,k} &= v \sum_{i=s}^{k} \binom{k}{i} \int_o^1 (1-t)^i t^{k+v-i-1} dt \\ &= v \sum_{i=s}^{k} \binom{k}{i} B\left(i+1,\ k+v-i\right) \\ &= v \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left(\prod_{j=0}^{i} (k+v-j)\right)^{-1}. \end{aligned} \tag{4}$$

2.1 E-Bayesian estimation of $R_{s,k}$

In this subsection, we calculate the Bayesian and E-Bayesian estimates for $R_{s,k}$. We assume that θ_1 and θ_2 have independent $gamma(a, b)$ and $gamma(c, d)$ priors, respectively. for $a, b, c, d > 0$, i.e.,

$$\begin{aligned} \pi_1(\theta_1|\theta_1 a, b) &= \frac{b^a}{\Gamma(a)} \theta_1^{a-1} e^{-b\theta_1}, \\ \pi_2(\theta_2|c, d) &= \frac{d^c}{\Gamma(c)} \theta_2^{c-1} e^{-d\theta_2}. \end{aligned}$$

The derivative of $\pi_1(\theta_1|\theta_1 a, b)$ with respect to θ_1 is given by

$$\frac{d\pi_1(\theta_1|\theta_1 a, b)}{d\theta_1} = \frac{b^a {\theta_1}^{a-2} e^{-b\theta_1}}{\Gamma(a)}((a-1) - b\theta_1).$$

According to Han (1997), it is recommended to choose “a” and “b” in a way that ensures $\pi_1(\theta_1|\theta_1 a, b)$is a decreasing function of θ_1. Therefore, “b” should be greater than 0 and “a” should be between 0 and 1. When $a = 1$, increasing the value of b results in a thinner tail of the density function. However, a thinner-tailed prior distribution often reduces the robustness of Bayesian estimation. Hence, the hyper-parameter b should be chosen such that it satisfies the constraint $0 < b < c_1$, where c_1 is a specified upper bound (a positive constant). For this study, we focus exclusively on the case when $a = 1$.

In this case, the density function $\pi_1(\theta_1|\theta_1 a, b)$ becomes

$$\pi_1\left(\theta_1|b\right) = be^{-b\theta_1}, \qquad \theta_1 > 0. \tag{5}$$

Also, the derivative of $\pi_2\left(\theta_2|c,d\right)$ with respect to θ_2 is given by

$$\frac{d\pi_2\left(\theta_2|c,d\right)}{d\theta_2} = \frac{d^c{\theta_2}^{c-2}e^{-d\theta_2}}{\Gamma(c)}\left((c-1)-d\theta_2\right).$$

Similarly, in accordance with Han (1997), c and d should be chosen in such way to guarantee that $\pi_2(\theta_2|c,d)$ is a decreasing function of θ_2. Thus, $d > 0$ and $0 < c < 1$. Given $c = 1$, and the larger the value of d, the thinner the tail of the density function. The hyper-parameter d should be chosen under the restriction $0 < d < c_2$, where c_2 is a given upper bound (c_2 is a positive constant). In this study, we only consider the case when $c = 1$. In this case, the density function $\pi_2\left(\theta_2|c,d\right)$ becomes

$$\pi_2\left(\theta_2 d\right) = de^{-e\theta_2}, \qquad \theta_2 > 0. \tag{6}$$

Based on the priors (5) and (6), the joint prior of θ_1 and θ_2 is

$$\pi\left(\theta_1,\theta_2\right) = bde^{-b\theta_1-d\theta_2}, \theta_1 > 0, \theta_2 > 0, \qquad b,d > 0. \tag{7}$$

Also, hyper-parameters b and d satisfy $D = \{(b,d)|0 < b < c_1,\ 0 < d < c_2\}$. Suppose that the prior distribution of b is uniform distribution in $(0, c_1)$, and the prior distribution of d is uniform distribution in $(0, c_2)$, when b and d are independent, the joint density of b and d is given by

$$\pi\left(b,d\right) = \pi\left(b\right)\pi\left(d\right) = \frac{1}{c_1c_2}, \qquad 0 < b < c_1, 0 < d < c_2.$$

Suppose $X_1,\ldots,X_n$ is a random sample of $IR(\theta_1)$ and $Y_1,\ldots,Y_m$ is a random of $IR\left(\theta_2\right)$. Therefore, the likelihood function of the observed data can be written as

$$L\left(\text{data}|\theta_1,\theta_2\right) \propto \theta_1^n\theta_2^m \exp\left(-\theta_1\sum_{i=1}^{n}\frac{1}{x_i^2} - \theta_2\sum_{j=1}^{m}\frac{1}{y_j^2}\right), \tag{8}$$

where

$$s_1 = b + \sum_{i=1}^{n}\frac{1}{x_i^2}, \qquad s_2 = e + \sum_{j=1}^{m}\frac{1}{y_j^2}.$$

The Bayesian estimation of $R_{s,k}$, under the SEL function is

$$\hat{R}_{Bay}\left(b,d\right) = \sum_{i=s}^{k}\frac{k!}{(k-i)!}\int_0^{\infty}\int_0^{\infty} v\left(\prod_{j=0}^{i}\left(k+v-j\right)\right)^{-1}\pi^*\left(\theta_1,\theta_2|\text{data}\right)d\theta_1 d\theta_2$$

$$=\sum_{i=s}^{k}\frac{k!}{(k-i)!}\int_0^{\infty}\int_0^{\infty}\left(\frac{\theta_2}{\theta_1}\right)\prod_{j=0}^{i}\left(\frac{\theta_2}{\theta_2+\theta_1(k-j)}\right)\pi^*(\theta_1,\theta_2|\text{data})\,d\theta_1 d\theta_2.$$

Since obtaining a closed form expression for $\hat{R}_{s,k}$ is impossible, we can expand $\frac{\theta_2}{\theta_2+\theta_1(k-j)}$ also in Taylor series (ignoring powers above 2), and approximate $\hat{R}_{Bay}(b,d)$. Therefore,

$$\begin{aligned}\frac{\theta_2}{\theta_2+\theta_1(k-j)} \approx\ & \frac{1}{k+1-j}+\frac{1}{(k+1-j)^2}(\theta_1-1)-\frac{1}{(k+1-j)^2}(\theta_2-1)\\ & -\frac{k-j}{(k+1-j)^3}(\theta_1-1)^2+\frac{1}{(k+1-j)^3}(\theta_2-1)^2\\ & +\frac{k-j-1}{(k+1-j)^3}(\theta_1-1)(\theta_2-1).\end{aligned} \tag{9}$$

According to (9), we can obtain $\hat{R}_{Bay}(b,d)$ as the following form

$$\begin{aligned}\hat{R}_{Bay}(b,d)\approx&\sum_{i=s}^{k}\frac{k!}{(k-i)!}\prod_{j=0}^{i}\Bigg[\frac{(k+1-j)A_{02}+A_{12}-A_{03}}{(k+1-j)^2}\\ &+\frac{(k-j-3)A_{03}-(k+1-j)A_{12}+(k-j-1)A_{13}+A_{14}-(k-j)A_{22}}{(k+1-j)^3}\Bigg],\end{aligned}$$

where $A_{sl}=\frac{\Gamma(n+s)\Gamma(m+l)}{s_1^{n+s}s_2^{m+l}}$. The definition for E-Bayesian estimation was originally addressed by Han (2006) as follows.

Definition 2.1. *In consideration prior of* $\hat{R}_{Bay}(b,d)$*,*

$$\hat{R}_{EB}=\int\int\hat{R}_{Bay}(b,d)\pi_3(b,d)\,dbdd, \qquad b,d\in D,$$

is called the E-Bayesian estimation of $R_{s,k}$ (briefly E-Bayesian estimation, the full name should be expected Bayesian estimation), which is assumed to be finite, where D is the domain of b and d, $\hat{R}_{Bay}(b,d)$ is the Bayesian estimation of $R_{s,k}$ with hyper parameters b and d, and $\pi_3(b,d)$ is the density function of b and d over D.

Definition (2.1) indicates that the E-Bayesian estimation of $R_{s,k}$ is just the expectation of the Bayesian estimation of $R_{s,k}$ for all the hyperparameters. Therefore, according to Equations (9), the E-Bayesian estimation of $R_{s,k}$ is given by

$$\begin{aligned}\hat{R}_{EB}=&\frac{1}{c_1c_2}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\prod_{j=0}^{i}\Bigg[\frac{(k+1-j)B_0C_2+B_1C_2-B_0C_3}{(k+1-j)^2}\\ &+\frac{(k-j-3)B_0C_3-(k+1-j)B_1C_2+(k-j-1)B_1C_3+B_1C_4-(k-j)B_2C_2}{(k+1-j)^3}\Bigg],\end{aligned}$$

where

$$B_r = (n+r)\Gamma(n+r)\left[\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n+r+1)}-\left(c_1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n+r+1)}\right],$$

$$C_r = (m+r)\,\Gamma(m+r)\left[\left(\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-(m+r+1)} - \left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-(m+r+1)}\right].$$

Lindley and Smith (1972) addressed an idea of a hierarchical prior distribution, as follows.

Definition 2.2. *If λ is hyperparameter in the parameter of θ, the prior density function of θ is $\pi(\theta|\lambda)$, and the prior density function of the hyperparameter of λ is $\pi(\lambda)$, then the hierarchical prior density function of θ is defined as follows:*

$$\pi(\theta) = \int \pi(\theta|\lambda)\pi(\lambda)\,d\lambda, \qquad \lambda\in\Lambda.$$

According to equations (5) and (6) and definition (2.2), the hierarchical prior density distributions of θ_1 and θ_2 are given by

$$\pi(\theta_i) = \frac{1-c_i\theta_i e^{-c_i\theta_i}-e^{-c_i\theta_i}}{c_i\theta_i^2}, \qquad i=1,2. \tag{10}$$

According to equations (8) and (10), the hierarchical posterior density function of θ_1 and θ_2 given the data is

$$\begin{aligned}\pi^{**}(\theta_1,\theta_2|\text{data}) \;\propto\; & \theta_1^{n-2}\theta_2^{m-2}e^{-c_1\theta_1-c_2\theta_2}(1-c_1\theta_1e^{-c_1\theta_1}-e^{-c_1\theta_1})\\ & \times(1-c_2\theta_2e^{-c_2\theta_2}-e^{-c_2\theta_2}).\end{aligned} \tag{11}$$

Now, using equations (9) and (11), the hierarchical Bayesian estimation of $\mathrm{R_{s,k}}$ under the SEL function is as follows:

$$\begin{aligned}\hat{R}_{HB} &= \sum_{i=s}^{k}\frac{k!}{(k-i)!}\int_0^\infty\int_0^\infty\left(\frac{\theta_2}{\theta_1}\right)\prod_{j=0}^{i}\left(\frac{\theta_2}{\theta_2+\theta_1(k-j)}\right)\pi^{**}(\theta_1,\theta_2|\text{data})\,d\theta_1 d\theta_2\\ &\approx \sum_{i=s}^{k}\frac{k!}{(k-i)!}\prod_{j=0}^{i}\frac{1}{k+1-j}S(n,m)+\frac{3k-2j+2}{(k+1-j)^3}S(n+1,m)\\ &\quad -\frac{2}{(k+1-j)^2}S(n,m+1)-\frac{k-j}{(k+1-j)^3}S(n+2,m)\\ &\quad +\frac{1}{(k+1-j)^3}S(n,m+2)+\frac{k-j-1}{(k+1-j)^3}S(n+1,m+1),\end{aligned}$$

where

$$\begin{aligned}S(n,m) &= \Gamma(n-2)\,\Gamma(m)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-2)}\left(\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-m}\\ &\quad -c_2\Gamma(n-2)\,\Gamma(m+1)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-2)}\left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-(m+1)}\end{aligned}$$

$$-\Gamma(n-2)\,\Gamma(m-1)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-2)}\left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-(m-1)}$$

$$-c_1\Gamma(n-1)\,\Gamma(m)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-1)}\left(\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-m}$$

$$-\Gamma(n-2)\,\Gamma(m)\left(c_1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-2)}\left(\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-m}$$

$$+c_1c_2\Gamma(n-1)\,\Gamma(m+1)\left(c_1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-1)}\left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-(m+1)}$$

$$+c_1\Gamma(n-1)\,\Gamma(m)\left(c_1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-1)}\left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-m}$$

$$+c_2\Gamma(n-2)\,\Gamma(m+1)\left(c_1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-2)}\left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-(m+1)}$$

$$+\Gamma(n-2)\,\Gamma(m)\left(c_1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{-(n-2)}\left(c_2+\sum_{j=1}^{m}\frac{1}{y_j^2}\right)^{-m}.$$

3 Numerical experiments

In this section, a Monte Carlo simulation is presented to illustrate all the estimation methods described in Section 2.

3.1 Simulation study

In this subsection, the Bayesian estimation, E-Bayesian, and hierarchical Bayesian of parameter $R_{s,k}$ are compared together. The simulation steps are shown below.
Step 1: For given value of the prior parameter $(0, c_1)$ we generate b from the uniform prior density $\pi(b) = \frac{1}{c_1}$, $0 < b < c_1$.
Step 2: For given value of the prior parameter $(0, c_2)$ we generate d from the uniform prior density $\pi(d) = \frac{1}{c_2}$, $0 < d < c_2$.
Step 3: θ_1 is produced using b estimated in the step 1, using equation (5), and θ_2 is produced using d estimated in the step 2, using equation (6).
Step 4: For given value of the θ_1 and θ_2, the samples with different n and m of IR(θ_1) and IR(θ_2) distributions, respectively are produced.
Step 5: Using the θ_1 and θ_2 estimated in the step 4, the samples with different n and m of IR(θ_1) and IR(θ_2) distributions, respectively. Then, the Bayesian, E-Bayesian, and hierarchical Bayesian estimations of $\mathrm{R_{s,k}}$ were estimated.

Steps 1 to 5 have been repeated 1000 times, and the average absolute bias (AAB)

Table 1: The AAB and MSE of the estimates of $R_{s,k}$. In each cell the second row represents MSE of the estimates of $R_{s,k}$.

(s,k)	(n,m)	$\hat{R}_B$	$\hat{R}_{EB}$	$\hat{R}_{HB}$
		$(c_1,\ c_2)=(3,4)$		
(2,4)	(10,10)	0.1552	0.1988	0.1432
		(0.0174)	(0.1277)	(0.0156)
	(10, 20)	0.1466	0.1877	0.1355
		(0.0167)	(0.1174)	(0.0140)
	(10, 30)	0.1323	0.1790	0.1290
		(0.0152)	(0.1168)	(0.0138)
	(10,50)	0.1299	0.1653	0.1199
		(0.0145)	(0.1159)	(0.0127)
	(20,20)	0.1179	0.1568	0.1088
		(0.0137)	(0.1143)	(0.0118)
	(30,20)	0.1099	0.1425	0.0977
		(0.0129)	(0.1139)	(0.0108)
	(50,20)	0.0988	0.1377	0.0879
		(0.0118)	(0.1129)	(0.0097)
(3,5)	(10,10)	0.1444	0.1766	0.1299
		(0.0157)	(0.1168)	(0.0138)
	(10,20)	0.1337	0.1658	0.1199
		(0.0148)	(0.1154)	(0.0123)
	(10,30)	0.1298	0.1588	0.1190
		(0.0139)	(0.1145)	(0.0118)
	(10,50)	0.1175	0.1433	0.1078
		(0.0129)	(0.1138)	(0.0108)
	(20,20)	0.1169	0.1389	0.1045
		(0.0117)	(0.1129)	(0.0099)
	(30, 20)	0.1159	0.1299	0.0943
		(0.0102)	(0.1101)	(0.0088)
	(50, 20)	0.1150	0.1219	0.0765
		(0.0100)	(0.1010)	(0.0068)
		$(c_1,\ c_2)=(3.5,4.5)$		
(2,4)	(20,30)	0.1434	0.1757	0.1290
		(0.0159)	(0.1168)	(0.0129)
	(50, 40)	0.1319	0.1646	0.1200
		(0.0147)	(0.1157)	(0.0117)
	(70, 50)	0.1208	0.1548	0.1119
		(0.0135)	(0.01147)	(0.0109)
	(80, 90)	0.1199	0.1477	0.1109
		(0.0127)	(0.0109)	(0.0099)
(3,5)	(20, 30)	0.1379	0.1623	0.1177
		(0.0128)	(0.1128)	(0.0117)
	(50, 40)	0.1225	0.1524	0.1158
		(0.0118)	(0.1009)	(0.0107)
	(70, 50)	0.1169	0.1433	0.1149
		(0.0092)	(0.0924)	(0.0088)
	(80, 90)	0.1157	0.1246	0.1127
		(0.0083)	(0.0743)	(0.0055)

estimation and its mean square error (MSE) were estimated and are listed in Table 1. All estimates under the SEL function is obtained for $(s,k)=(2,4),(3,\ 5)$ and $(c_1,\ c_2)=(3,4),(3.5,4.5)$, respectively. The performance of all estimates have been

compared numerically of the MSE value. The simulation is conducted by R software. Based on tabulated ABB and MSE values, the following conclusions can be drawn from Table 1.
a. Since ABB and MSE values of the hierarchical Bayesian estimation of $R_{s,k}$ are less than ABB and MSE values of those of Bayesian and E-Bayesian estimations in both cases of (s,k), therefore, the performance of the hierarchical Bayesian estimation of $R_{s,k}$ under SEL function is better than that of Bayesian and E-Bayesian estimations. Also, when n and m are increase, the MSE of all estimators decreases.
b. Since ABB and MSE values of all estimators in state $(s,k) = (3,5)$ are less than ABB and MSE of all estimators in state $(s,k) = (2,4)$, for both cases $(c_1,\ c_2)$, we conclude that three out of five component system reliability is more than the two out of four component system.

4 Application with real data set

In this subsection, we present a data analysis of the strength data reported by Badar and Priest (1982). This data, represent the strength measured in GPA for single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20mm (Data Set 1) and 10mm (Data Set 2).

Since for Data Set 1, the value of the Kolmogorov-Smirnov (K-S) statistic is 0.0815, with a corresponding p-value of 0.9197, the data follows the IR distribution. Also, we have for Data Set 2, the K-S statistic equal to 0.133, with a corresponding p-value of 0.8103. So, the data follows the IR distribution.

To compute the Bayesian, E-Bayesian, and hierarchical Bayesian estimations, since we do not have any prior information, we assumed that$b = d = 0.001$. For $c_1 = c_2 = 3$, $\theta_1 = 1.5$, $\theta_2 = 2.5$, and for $(s,k) = (2,\ 4)$, the Bayesian, E-Bayesian, and hierarchical Bayesian estimations become 0.767789, 0.678907 and 0.876226, respectively, Also, for $(s,k) = (3,5)$, the Bayesian, E-Bayesian, and hierarchical Bayesian estimations become 0.873342, 0.798765 and 0.945321, respectively. Because the value of the hierarchical Bayesian estimation of reliability in a multicomponent stress-strength system is greater than the value of the reliability estimation of other estimation methods in both the cases of (s,k), therefore, the performance of the hierarchical Bayesian estimation of $R_{s,k}$ is better than that of Bayesian and E-Bayesian estimations. We can see, when $(s,k) = (3,5)$, the value of the hierarchical Bayesian estimation of the $R_{s,k}$ estimation is greater than the value of the hierarchical Bayesian estimation of the $R_{s,k}$ estimation in the case $(s,k) = (2,4)$, we conclude that three out of five component system reliability is more than the two out of four component system reliability.

5 Conclusions

In this study, Bayesian, E-Bayesian, and hierarchical Bayesian estimations of reliability in a multicomponent stress-strength model, were obtained. We assume that the underlying distribution for both stress and strength variables have IR distributions with different scale parameters. By calculating the MSE and the average absolute bias estimation, the Bayesian, E-Bayesian, and hierarchical Bayesian estimations of reliability in a multicomponent stress-strength model based on the IR distribution were compared using Monte Carlo simulation and two real data sets. It has been shown that the estimation of reliability by hierarchical Bayesian estimation has better efficiency. Furthermore, it was shown that the reliability of the one out of three component system is higher than the reliability of the one out of two component system for three estimation methods.

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