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Research Paper

Estimations of the parameters for modified Weibull distribution under adaptive type-II progressive censored samples

Akram Kohansal*, Hasan Haji
Department of Statistics, Imam Khomeini International University, Qazvin, Iran

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Abstract: This paper describes the point and interval estimation of the unknown parameters of modified Weibull distribution under the adaptive Type-II progressive censored samples. First, we obtain the maximum likelihood estimation of parameters. Because maximum likelihood estimations should be solved in numerical methods and cannot be derived in a closed form, the approximate maximum likelihood estimations of the parameters are achieved. Also, asymptotic confidence intervals are obtained by earning the asymptotic distribution of the parameters. Moreover, two bootstrap confidence intervals are derived. Second, the Bayesian estimation of parameters is approximated using the Markov chain Monte Carlo algorithm and Lindley's method. Furthermore, the highest posterior density credible intervals of the parameters are derived. Finally, the different proposed estimations have been compared by the simulation studies and one data set is analyzed to illustrative aims.

Keywords: Adaptive Type-II progressive censored samples; Approximate maximum likelihood estimation; Markov chain Monte Carlo algorithm; Modified Weibull distribution.
Mathematics Subject Classification (2010): 62F10, 62F15.

1 Introduction

Among different censoring schemes, Type-I and Type-II are two most fundamental schemes. The test is finished in a pre-determined time and in a pre-chosen number of failures, in Type-I and Type-II censoring schemes, respectively. Hybrid censoring scheme which is earned by mixing Type-I and Type-II schemes (see Epstein, 1954),

*Corresponding author: kohansal@sci.ikiu.ac.ir

finished the test at time $T^* = \min\{X_{m:n}, T\}$, where $X_{m:n}$ is the m-th failure times from n items and $T > 0$. Because none of the above schemes allows the removal of active units during the test, Progressive censoring scheme is introduced. Combining Type-II and progressive schemes, Type-II progressive censoring scheme is provided. Also, by mixing the progressive and hybrid scheme, the hybrid progressive scheme is derived. The Type-II hybrid progressive scheme which has been initiated by Kundu and Joarder (2006), can be explained as follows: Suppose N units are put on the test with the censoring scheme $(R_1, \ldots, R_n)$ and halting time $T^* = \min\{X_{n:n:N}, T\}$, where $X_{1:n:N} \leq \ldots \leq X_{n:n:N}$ come from a progressive censoring scheme and $T > 0$ is fixed. It is obvious that the experiment stop time is $X_{n:n:N}$ if $X_{n:n:N} < T$ and T if $X_{J:n:N} < T < X_{J+1:n:N}$. In the first case n failures are happen and in second case J failures are happen. As we see, in this scheme, the sample size is random and it is very likely that we confront a very small number of units. So, in many practical situation, the statistical inference cannot be applicable. Therefore, a new scheme is introduced by Ng et al. (2009) which is named as adaptive progressive scheme. The adaptive Type-II progressive censoring (AT-II PC) scheme can be explained as follows: Suppose $X_{1:n:N}, \ldots, X_{n:n:N}$ be a progressive censoring sample and $T > 0$ is fixed. It is obvious that the experiment stop time is $X_{n:n:N}$ if $X_{n:n:N} < T$ and a Type-II progressive censoring sample with the progressive censoring scheme $R_1, \ldots, R_n$ are happen. Otherwise, if $X_{J:n:N} < T < X_{J+1:n:N}$ where $J + 1 < n$, then we will not withdraw any items from the experiment by setting

$$R_{J+1} = \ldots = R_{n-1} = 0,\ R_n = N - n - \sum_{i=1}^{J} R_i.$$

Herein, an AT-II PC sample will be denoted with $\{X_1, \ldots, X_n\}$ under the scheme $\{N, n, T, J, R_1, \ldots R_n\}$ such that $X_J < T < X_{J+1}$. Assuring to get the fix number of units n is one of the most advantage of this scheme. Furthermore, this scheme is general, in that if $T = 0$ and $T = \infty$, the Type-II and Type-II progressive censoring schemes cab be derived from it, respectively. Very recently Nassar and Abo-Kasem (2017) studied the estimation of the inverse Weibull parameters under adaptive type-II progressive censoring scheme. In this paper, under the AT-II PC scheme, we estimate the unknown parameters of the modified Weibull distribution.

A random variable X is said to have modified Weibull (mW) distribution with the parameters α, β and λ in symbols $X \sim mW(\alpha, \beta, \lambda)$, if its probability density function (pdf), cumulative distribution function (cdf) and failure rate function, respectively are

$$f_X(x) = \alpha(\beta + \lambda x)x^{\beta-1}e^{\lambda x}e^{-\alpha x^{\beta} e^{\lambda x}},\ \ x > 0,\ \alpha, \beta, \lambda > 0, \tag{1}$$

$$F_X(x) = 1 - e^{-\alpha x^{\beta} e^{\lambda x}}, \tag{2}$$

$$H(x) = \alpha(\beta + \lambda x)x^{\beta-1}e^{\lambda x}, \tag{3}$$

The pdf and failure rate function of mW distribution are plotted in Figure 1. As we see the failure rate function of mW distribution is an increasing and decreasing function. So if the empirical study proposes that the failure rate function of the prior distribution is increasing or decreasing, then the mW distribution can be used to analyze the data set. This paper is organized as follows: In the next section the classical point

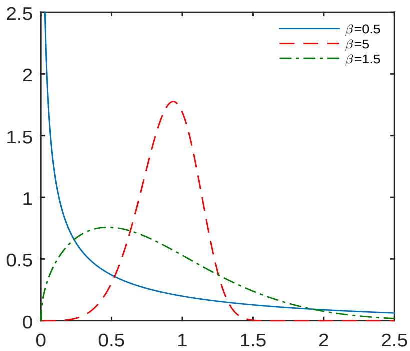

Figure 1: Shape of pdf (left) and failure rate function (right) of mW distribution when $\alpha = 1$, $\lambda = 0.1$.

and interval estimation are derived. So, the maximum likelihood estimation (MLE) of the unknown parameters are obtained. As we will see the obtained estimations are not closed forms and we should solve the numerical equation. So we approximate the MLEs and derived approximate maximum likelihood estimations (AMLE). Also, by providing the asymptotic distribution of unknown parameters, we construct the asymptotic confidence intervals for the parameters. Because the asymptotic confidence intervals may not be appropriate when the sample size is small, two bootstrap confidence intervals i.e. Boot-t and Boot-p are achieved. In section 3, the Bayes estimates of the parameters are obtained by using Lindley's approximation and the Markov chain Monte Carlo (MCMC) method when the parameters α, β and λ have statistically independent gamma priors. The highest posterior density (HPD) credible interval for the unknown parameters are constructed in this section.The simulation results and data analysis are reported in section 4.

2 Classical point and interval estimation

2.1 Maximum likelihood estimation

In this section, MLE of the unknown parameters are derived. Let $\{X_1, \ldots, X_n\}$ be an AT-II PC sample from $mW(\alpha, \beta, \lambda)$ under the scheme $\{N, n, T, J, R_1, \ldots, R_n\}$ such that $X_J < T < X_{J+1}$. The likelihood function of the parameters is derived as

$$L(\alpha, \beta, \lambda) = c \prod_{i=1}^{n} f(x_i) \prod_{i=1}^{J} \big(1 - F(x_i)\big)^{R_i} \big(1 - F(x_n)\big)^{R_n},$$

where $c = \prod_{i=1}^{n} \Big(N - i + 1 - \sum_{j=1}^{\min\{i-1,J\}} R_j\Big)$, $R_n = N - n - \sum_{i=1}^{J} R_i$. Based on the observed data, $\underset{\sim}{x}$, using the equations (1) and (2), the likelihood function is given by

$$L(\underset{\sim}{x}|\alpha, \beta, \lambda) \quad \propto \quad \alpha^n \big(\prod_{i=1}^{n}(\beta + \lambda x_i)\big) \times \big(\prod_{i=1}^{n} x_i^{\beta-1}\big) \times e^{\lambda \sum_{i=1}^{n} x_i}$$

$$\times e^{-\alpha\left(\sum\limits_{i=1}^{n} x_i^{\beta} e^{\lambda x_i}+\sum\limits_{i=1}^{J} R_i x_i^{\beta} e^{\lambda x_i}+R_n x_n^{\beta} e^{\lambda x_n}\right)}.$$

So, the log-likelihood function ignoring the constant value is

$$\begin{aligned}\ell(\alpha,\beta,\lambda) &= n\log(\alpha)+\sum_{i=1}^{n}\log(\beta+\lambda x_i)+(\beta-1)\sum_{i=1}^{n}\log(x_i)+\lambda\sum_{i=1}^{n}x_i\\ &\quad -\alpha\Big(\sum_{i=1}^{n}x_i^{\beta}e^{\lambda x_i}+\sum_{i=1}^{J}R_i x_i^{\beta}e^{\lambda x_i}+R_n x_n^{\beta}e^{\lambda x_n}\Big). \qquad (4)\end{aligned}$$

Therefore, $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$, the MLEs of α, β and λ, respectively, can be derived by solving the following equations

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha}-\sum_{i=1}^{n}x_i^{\beta}e^{\lambda x_i}-\sum_{i=1}^{J}R_i x_i^{\beta}e^{\lambda x_i}-R_n x_n^{\beta}e^{\lambda x_n}=0, \qquad (5)$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} &= \sum_{i=1}^{n}\frac{1}{\beta+\lambda x_i}+\sum_{i=1}^{n}\log(x_i)-\alpha\Big(\sum_{i=1}^{n}x_i^{\beta}\log(x_i)e^{\lambda x_i}\\ &\quad +\sum_{i=1}^{J}R_i x_i^{\beta}\log(x_i)e^{\lambda x_i}+R_n x_n^{\beta}\log(x_n)e^{\lambda x_n}\Big)=0, \qquad (6)\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} &= \sum_{i=1}^{n}\frac{x_i}{\beta+\lambda x_i}+\sum_{i=1}^{n}x_i-\alpha\Big(\sum_{i=1}^{n}x_i^{\beta+1}e^{\lambda x_i}\\ &\quad +\sum_{i=1}^{J}R_i x_i^{\beta+1}e^{\lambda x_i}+R_n x_n^{\beta+1}e^{\lambda x_n}\Big)=0. \qquad (7)\end{aligned}$$

So, from (5), $\widehat{\alpha}(\beta,\lambda)=n\Big\{\sum\limits_{i=1}^{n}x_i^{\beta}e^{\lambda x_i}+\sum\limits_{i=1}^{J}R_i x_i^{\beta}e^{\lambda x_i}+R_n x_n^{\beta}e^{\lambda x_n}\Big\}^{-1}$. Also, using a numerical method such as Newton-Raphson on the non-linear equations in equations (6) and (7), the MLEs can be derived.

2.2 Approximate maximum likelihood estimation

As the MLEs of the parameters cannot be obtained in the closed form, we derive AMLEs of the parameters, which have closed forms.

Lemma 2.1. *Let $X \sim mW(\alpha,\beta,\lambda)$.*
(i) If $Y_1 \overset{d}{=} X$ with $\lambda=0$, then $Y_1 \sim W(\beta,\alpha)$ with the following cdf

$$F_{Y_1}(y)=1-e^{-\alpha y^{\beta}}, \qquad y>0,\ \alpha,\beta>0.$$

(ii) If $Y_2 \overset{d}{=} \log(Y_1)$, then $Y_2 \sim EV(\mu,\sigma)$ with the following cdf

$$F_{Y_2}(y)=1-e^{-e^{\frac{y-\mu}{\sigma}}}, \quad y\in\mathbb{R},$$

where $\mu=-\frac{1}{\beta}\log(\alpha)$ and $\sigma=\frac{1}{\beta}$.

Let $\{X_1,\ldots,X_n\}$ be an AT-II PC sample from $mW(\alpha,\beta,\lambda)$ under the scheme $\{N,n,T,J,R_1,\ldots,R_n\}$ such that $X_J < T < X_{J+1}$. Moreover, let $Y_i \overset{d}{=} X_i$ $(\lambda=0)$, $U_i \overset{d}{=} \log(Y_i)$. Now, using Lemma 2.1, $U_i \sim EV(\mu,\sigma)$, where $\mu = -\frac{1}{\beta}\log(\alpha),\ \sigma=\frac{1}{\beta}$. Therefore, the log-likelihood function of the observed data $\{U_1,\ldots,U_n\}$, ignoring the constant value, is

$$\ell^*(\mu,\sigma)=\sum_{i=1}^{n} t_i-\sum_{i=1}^{n} e^{t_i}-\sum_{i=1}^{J} R_i e^{t_i}-R_n e^{t_n}-n\log(\sigma),$$

where $t_i=\frac{u_i-\mu}{\sigma}$. By taking derivatives of the above equation with respect to μ and σ we have

$$\begin{aligned}\frac{\partial \ell^*}{\partial \mu} &= -\frac{1}{\sigma}\left[n-\sum_{i=1}^{n} e^{t_i}-\sum_{i=1}^{J_1} R_i e^{t_i}-R_n e^{t_n}\right]=0,\\ \frac{\partial \ell^*}{\partial \sigma} &= -\frac{1}{\sigma}\left[n+\sum_{i=1}^{n} t_i-\sum_{i=1}^{n} t_i e^{t_i}-\sum_{i=1}^{n} R_i t_i e^{t_i}-R_n t_n e^{t_n}\right]=0.\end{aligned}$$

To derive the AMLEs of α and β, we expand the functions e^{t_i} in Taylor series around the point $\nu_i=\log\big(-\log(1-q_i)\big)$, where

$$q_i = 1-\prod_{j=n-i+1}^{n}\frac{j+\sum\limits_{k=n-j+1}^{n} R_k}{j+1+\sum\limits_{k=n-j+1}^{n} R_k},\ i=1,\ldots,n.$$

By keeping the first order derivatives: $e^{t_i}=\alpha_i+\beta_i t_i$, where $\alpha_i=e^{\nu_i}(1-\nu_i),\ \beta_i=e^{\nu_i}$. Using the linear approximations, we obtain the AMLEs of μ and σ, say $\tilde{\mu}$ and $\tilde{\sigma}$, respectively, by

$$\tilde{\mu}=A+\tilde{\sigma}B, \qquad \tilde{\sigma}=\frac{-D+\sqrt{(D^2+4CE)}}{2C},$$

where

$$\begin{aligned} A &= \frac{\sum\limits_{i=1}^{n}\beta_i u_i+\sum\limits_{i=1}^{J} R_i\beta_i u_i+R_n\beta_n u_n}{\sum\limits_{i=1}^{n}\beta_i+\sum\limits_{i=1}^{J} R_i\beta_i+R_n\beta_n}, \qquad B=\frac{-n+\sum\limits_{i=1}^{n}\alpha_i+\sum\limits_{i=1}^{J} R_i\alpha_i+R_n\alpha_n}{\sum\limits_{i=1}^{n}\beta_i+\sum\limits_{i=1}^{J} R_i\beta_i+R_n\beta_n},\\ C &= n-nB+B(\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{J_1} R_i\alpha_i+R_n\alpha_n)-B^2(\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{J}\beta_i R_i-R_n\beta_n),\\ D &= \sum_{i=1}^{n}(u_i-A)-\sum_{i=1}^{n}\alpha_i(u_i-A)-\sum_{i=1}^{J} R_i\alpha_i(u_i-A)-R_n\alpha_n(u_n-A)\\ &\quad +2B(\sum_{i=1}^{n}\beta_i(u_i-A)+\sum_{i=1}^{J} R_i\beta_i(u_i-A)+R_n\beta_n(u_n-A)),\end{aligned}$$

$$E = \sum_{i=1}^{n}\beta_i(u_i-A)^2+\sum_{i=1}^{J}R_i\beta_i(u_i-A)^2+R_n\beta_n(u_n-A)^2.$$

After obtaining $\tilde{\mu}$ and $\tilde{\sigma}$, the AMLEs of α and β, say $\tilde{\alpha}$ and $\tilde{\beta}$, respectively, can be evaluated by

$$\tilde{\alpha}=e^{-\frac{\tilde{\mu}}{\tilde{\sigma}}},\quad \tilde{\beta}=\frac{1}{\tilde{\sigma}}. \tag{8}$$

2.3 Asymptotic confidence interval

In this section, by deriving the asymptotic distribution of $\widehat{\theta}=(\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$, the asymptotic confidence intervals can be obtained. If $I(\theta)=[I_{ij}]=\left[-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\right]$, $i,j=1,2,3$, is the observed information matrix, then, from the log-likelihood function in equation (4), we have

$$\begin{aligned}
I_{11} &= \frac{n}{\alpha^2}\\
I_{12} &= \sum_{i=1}^{n}x_i^{\beta}\log(x_i)e^{\lambda x_i}+\sum_{i=1}^{J}R_ix_i^{\beta}\log(x_i)e^{\lambda x_i}+R_nx_n^{\beta}\log(x_n)e^{\lambda x_n}=I_{21},\\
I_{13} &= \sum_{i=1}^{n}x_i^{\beta+1}e^{\lambda x_i}+\sum_{i=1}^{J}R_ix_i^{\beta+1}e^{\lambda x_i}+R_nx_n^{\beta+1}e^{\lambda x_n}=I_{31},\\
I_{22} &= \sum_{i=1}^{n}\frac{1}{(\beta+\lambda x_i)^2}+\alpha\big(\sum_{i=1}^{n}x_i^{\beta}\log^2(x_i)e^{\lambda x_i}\\
&\quad+\sum_{i=1}^{J}R_ix_i^{\beta}\log^2(x_i)e^{\lambda x_i}+R_nx_n^{\beta}\log^2(x_n)e^{\lambda x_n}\big),\\
I_{23} &= \sum_{i=1}^{n}\frac{x_i}{(\beta+\lambda x_i)^2}+\alpha\big(\sum_{i=1}^{n}x_i^{\beta+1}\log(x_i)e^{\lambda x_i}\\
&\quad+\sum_{i=1}^{J}R_ix_i^{\beta+1}\log(x_i)e^{\lambda x_i}+R_nx_n^{\beta+1}\log(x_n)e^{\lambda x_n}\big)=I_{32},\\
I_{33} &= \sum_{i=1}^{n}\big(\frac{x_i}{\beta+\lambda x_i}\big)^2+\alpha\big(\sum_{i=1}^{n}x_i^{\beta+2}e^{\lambda x_i}+\sum_{i=1}^{J}R_ix_i^{\beta+2}e^{\lambda x_i}+R_nx_n^{\beta+2}e^{\lambda x_n}\big).
\end{aligned}$$

From the asymptotic normality property of MLE, if $n\longrightarrow\infty$ then $[(\widehat{\alpha}-\alpha),\ (\widehat{\beta}-\beta),\ (\widehat{\lambda}-\lambda)]^T\overset{D}{\longrightarrow}N_3(0,\mathbf{I^{-1}}(\alpha,\beta,\lambda))$, where

$$\begin{aligned}
\mathbf{I}(\alpha,\beta,\lambda) &= \begin{bmatrix} I_{11} & I_{12} & I_{13}\\ & I_{22} & I_{23}\\ & & I_{33}\end{bmatrix},\\
\mathbf{I}^{-1}(\alpha,\beta,\lambda) &= \frac{1}{|I(\alpha,\beta,\lambda)|}\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ & b_{22} & b_{23}\\ & & b_{33}\end{bmatrix}=\begin{bmatrix} var(\widehat{\alpha}) & cov(\widehat{\alpha},\widehat{\beta}) & cov(\widehat{\alpha},\widehat{\lambda})\\ & var(\widehat{\beta}) & cov(\widehat{\beta},\widehat{\lambda})\\ & & var(\widehat{\lambda})\end{bmatrix},
\end{aligned}$$

where

$$
\begin{aligned}
|I(\alpha,\beta,\lambda)| &= I_{11}I_{22}I_{33}+2I_{12}I_{13}I_{23}-I_{11}I_{23}^2-I_{12}^2I_{33}-I_{13}^2I_{22},\\
b_{11} &= I_{22}I_{33}-I_{23}^2, \quad b_{12}=I_{13}I_{23}-I_{12}I_{33}, \quad b_{13}=I_{12}I_{23}-I_{13}I_{22},\\
b_{22} &= I_{11}I_{33}-I_{13}^2, \quad b_{23}=I_{12}I_{13}-I_{11}I_{23}, \quad b_{33}=I_{11}I_{22}-I_{12}^2.
\end{aligned}
$$

So, if z_η is 100η-th percentile of $N(0,1)$, $100(1-\eta)\%$ asymptotic confidence intervals of α, β and λ can be obtained respectively as

$$
\begin{aligned}
&(\widehat{\alpha}-z_{1-\frac{\eta}{2}}\sqrt{var(\widehat{\alpha})},\widehat{\alpha}+z_{1-\frac{\eta}{2}}\sqrt{var(\widehat{\alpha})}),\\
&(\widehat{\beta}-z_{1-\frac{\eta}{2}}\sqrt{var(\widehat{\beta})},\widehat{\beta}+z_{1-\frac{\eta}{2}}\sqrt{var(\widehat{\beta})}),\\
&(\widehat{\lambda}-z_{1-\frac{\eta}{2}}\sqrt{var(\widehat{\lambda})},\widehat{\lambda}+z_{1-\frac{\eta}{2}}\sqrt{var(\widehat{\lambda})}).
\end{aligned}
\tag{9}
$$

2.4 Bootstrap confidence interval

In this section, we utilize two bootstrap confidence intervals: Boot-p and Boot-t which proposed by Efron (1982) and Hall (1988), respectively.

Boot-p Method

1. Generate the AT-II PC samples $\{x_1,\dots,x_n\}$ with $\{N,n,T,J,R_1,\dots,R_n\}$ censoring scheme and estimate $(\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$.
2. Generate the AT-II PC samples $\{x_1^*,\dots,x_n^*\}$ with $\{N,n,T,J,R_1,\dots,R_n\}$ censoring scheme from $mW(\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$ and compute the MLE bootstrap estimate $(\widehat{\alpha}^*,\widehat{\beta}^*,\widehat{\lambda}^*)$.
3. Repeat step 2, NBOOT times.
4. Let $G_1^*(x_1)=P(\widehat{\alpha}^*\le x_1)$, $G_2^*(x_2)=P(\widehat{\beta}^*\le x_2)$ and $G_2^*(x_2)=P(\widehat{\lambda}^*\le x_3)$ be the cdf of $\widehat{\alpha}^*$, $\widehat{\beta}^*$ and $\widehat{\lambda}^*$, respectively. Define $\widehat{\alpha}_{Bp}(x_1)=G_1^{*-1}(x_1)$, $\widehat{\beta}_{Bp}(x_2)=G_2^{*-1}(x_2)$ and $\widehat{\lambda}_{Bp}(x_3)=G_3^{*-1}(x_3)$, for given x_1, x_2 and x_3, respectively. The $100(1-\eta)\%$ Boot-p confidence intervals of α, β and λ can be obtained respectively as

$$
(\widehat{\alpha}_{Bp}(\frac{\eta}{2}),\widehat{\alpha}_{Bp}(1-\frac{\eta}{2})),\quad (\widehat{\beta}_{Bp}(\frac{\eta}{2}),\widehat{\beta}_{Bp}(1-\frac{\eta}{2})),\quad (\widehat{\lambda}_{Bp}(\frac{\eta}{2}),\widehat{\lambda}_{Bp}(1-\frac{\eta}{2})). \tag{10}
$$

Boot-t Method

1. Generate the AT-II PC samples $\{x_1,\dots,x_n\}$ with $\{N,n,T,J,R_1,\dots,R_n\}$ censoring scheme and estimate $(\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$.
2. Generate the AT-II PC samples $\{x_1^*,\dots,x_n^*\}$ with $\{N,n,T,J,R_1,\dots,R_n\}$ censoring scheme from $mW(\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$ and compute the MLE bootstrap estimate $(\widehat{\alpha}^*,\widehat{\beta}^*,\widehat{\lambda^*})$. Using the previous section, calculate the statistics

$$
T_1^*=\frac{\widehat{\alpha}^*-\widehat{\alpha}}{\sqrt{var(\widehat{\alpha}^*)}},\quad T_2^*=\frac{\widehat{\beta}^*-\widehat{\beta}}{\sqrt{var(\widehat{\beta}^*)}},\quad T_3^*=\frac{\widehat{\lambda}^*-\widehat{\lambda}}{\sqrt{var(\widehat{\lambda}^*)}}.
$$

3. Repeat step 2, NBOOT times.
4. Let $H_1(x_1)=P(T_1^*\le x_1)$, $H_2(x_2)=P(T_2^*\le x_2)$ and $H_3(x_3)=P(T_3^*\le x_3)$ be the cdf of T_1^*, T_2^* and T_3^*, respectively. Define $\widehat{\alpha}_{Bt}(x_1)=\widehat{\alpha}+{H_1}^{-1}(x_1)\sqrt{var(\widehat{\alpha})}$, $\widehat{\beta}_{Bt}(x_2)=\widehat{\beta}+{H_2}^{-1}(x_2)\sqrt{var(\widehat{\beta})}$ and $\widehat{\lambda}_{Bt}(x_3)=\widehat{\lambda}+{H_3}^{-1}(x_3)\sqrt{var(\widehat{\lambda})}$, for given x_1,

x_2 and x_3, respectively. The $100(1-\eta)\%$ Boot-t confidence intervals of α, β and λ can be obtained respectively as

$$(\widehat{\alpha}_{Bt}(\frac{\eta}{2}), \widehat{\alpha}_{Bt}(1-\frac{\eta}{2})), \quad (\widehat{\beta}_{Bt}(\frac{\eta}{2}), \widehat{\beta}_{Bt}(1-\frac{\eta}{2})), \quad (\widehat{\lambda}_{Bt}(\frac{\eta}{2}), \widehat{\lambda}_{Bt}(1-\frac{\eta}{2})). \tag{11}$$

3 Bayesian inference

In this section, we cover Bayesian approaches to parameter estimation. For the prior distributions, we assumed that $\alpha \sim \Gamma(a_1, b_1)$, $\beta \sim \Gamma(a_2, b_2)$ and $\lambda \sim \Gamma(a_3, b_3)$ and they are independent. Based on the observed censoring samples, the joint posterior density function should be obtained as

$$\pi(\alpha, \beta, \lambda | \underset{\sim}{x}) \propto L(\underset{\sim}{x} | \alpha, \beta, \lambda) \pi_1(\alpha) \pi_2(\beta) \pi_3(\lambda), \tag{12}$$

where

$$\begin{aligned}
\pi_1(\alpha) &\propto \alpha^{a_1-1} e^{-b_1\alpha}, \ a_1, b_1 > 0, \\
\pi_2(\beta) &\propto \beta^{a_2-1} e^{-b_2\beta}, \ a_2, b_2 > 0, \\
\pi_3(\lambda) &\propto \lambda^{a_3-1} e^{-b_3\lambda}, \ a_3, b_3 > 0.
\end{aligned}$$

From the equation (12), it is observed that the Bayes estimation cannot be obtained in the closed form. So, we approximated the Bayes estimations by two method: Lindley's approximation and MCMC algorithm.

3.1 Lindley's approximation

One of the most numerical methods to derive the Bayesian estimation has been proposed by Lindley (1980). In this method, under the squared error loss function, if $U(\theta)$ is a function of the parameter value then the Bayes estimate of $U(\theta)$ is

$$\mathbb{E}\big(u(\theta) | \underset{\sim}{x}\big) = \frac{\int u(\theta) e^{Q(\theta)} d\theta}{\int e^{Q(\theta)} d\theta}, \tag{13}$$

where $Q(\theta) = \ell(\theta) + \rho(\theta)$, $\ell(\theta)$ is the logarithm of the likelihood function, and $\rho(\beta)$ is the logarithm of the prior density of θ. As stated by Lindley, we can approximate equation (13) as

$$\mathbb{E}(u(\theta)|\underset{\sim}{x}) = u + \frac{1}{2}\sum_i \sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_p \ell_{ijk}\sigma_{ij}\sigma_{kp}u_p \Big|_{\theta=\widehat{\theta}}, \tag{14}$$

where $\theta = (\theta_1, \ldots, \theta_m)$, $i, j, k, p = 1, \ldots, m$, $\widehat{\theta}$ is MLE of θ, $u = u(\theta)$, $u_i = \partial u / \partial \theta_i$, $u_{ij} = \partial^2 u / (\partial\theta_i \partial\theta_j)$, $\ell_{ijk} = \partial^3 \ell / (\partial\theta_i \partial\theta_j \partial\theta_k)$, $\rho_j = \partial\rho/\partial\theta_j$, and $\sigma_{ij} = (i,j)$-th element in inverse of matrix $[-\ell_{ij}]$ all evaluated at MLE of the parameters. So, simplifying equation (14), in the case of three parameters, $\theta = (\theta_1, \theta_2, \theta_3)$, we conclude that

$$\mathbb{E}(u(\theta)|\underset{\sim}{x}) \quad = \quad u + (u_1 d_1 + u_2 d_2 + u_3 d_3 + d_4 + d_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13})$$

$$+B(u_1\sigma_{12}+u_2\sigma_{22}+u_3\sigma_{23})+C(u_1\sigma_{13}+u_2\sigma_{23}+u_3\sigma_{33})], \qquad (15)$$

where

$$\begin{aligned}
d_i &= \rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3}, \quad i=1,2,3,\\
d_4 &= u_{12}\sigma_{12}+u_{13}\sigma_{13}+u_{23}\sigma_{23},\\
d_5 &= \frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22}+u_{33}\sigma_{33}),\\
A &= \ell_{111}\sigma_{11}+2\ell_{112}\sigma_{12}+2\ell_{113}\sigma_{13}+2\ell_{123}\sigma_{23}+\ell_{122}\sigma_{22}+\ell_{133}\sigma_{33},\\
B &= \ell_{112}\sigma_{11}+2\ell_{122}\sigma_{12}+2\ell_{123}\sigma_{13}+2\ell_{223}\sigma_{23}+\ell_{222}\sigma_{22}+\ell_{233}\sigma_{33},\\
C &= \ell_{113}\sigma_{11}+2\ell_{123}\sigma_{12}+2\ell_{133}\sigma_{13}+2\ell_{233}\sigma_{23}+\ell_{223}\sigma_{22}+\ell_{333}\sigma_{33}.
\end{aligned}$$

In our case, for $(\theta_1,\theta_2,\theta_3)\equiv(\alpha,\beta,\lambda)$, we have

$$\begin{aligned}
\rho_1 &= \frac{a_1-1}{\alpha}-b_1, \quad \rho_2=\frac{a_2-1}{\beta}-b_2, \quad \rho_3=\frac{a_3-1}{\lambda}-b_3,\\
\ell_{11} &= -\frac{n}{\alpha^2},\\
\ell_{12} &= -\sum_{i=1}^{n}x_i^{\beta}\log(x_i)e^{\lambda x_i}-\sum_{i=1}^{J}R_ix_i^{\beta}\log(x_i)e^{\lambda x_i}-R_nx_n^{\beta}\log(x_n)e^{\lambda x_n},\\
\ell_{13} &= -\sum_{i=1}^{n}x_i^{\beta+1}e^{\lambda x_i}-\sum_{i=1}^{J}R_ix_i^{\beta+1}e^{\lambda x_i}-R_nx_n^{\beta+1}e^{\lambda x_n}=I_{31},\\
\ell_{22} &= -\sum_{i=1}^{n}\frac{1}{(\beta+\lambda x_i)^2}-\alpha\Big(\sum_{i=1}^{n}x_i^{\beta}\log^2(x_i)e^{\lambda x_i}\\
&\quad+\sum_{i=1}^{J}R_ix_i^{\beta}\log^2(x_i)e^{\lambda x_i}+R_nx_n^{\beta}\log^2(x_n)e^{\lambda x_n}\Big),\\
\ell_{23} &= -\sum_{i=1}^{n}\frac{x_i}{(\beta+\lambda x_i)^2}-\alpha\Big(\sum_{i=1}^{n}x_i^{\beta+1}\log(x_i)e^{\lambda x_i}\\
&\quad+\sum_{i=1}^{J}R_ix_i^{\beta+1}\log(x_i)e^{\lambda x_i}+R_nx_n^{\beta+1}\log(x_n)e^{\lambda x_n}\Big),\\
\ell_{33} &= -\sum_{i=1}^{n}\Big(\frac{x_i}{\beta+\lambda x_i}\Big)^2-\alpha\Big(\sum_{i=1}^{n}x_i^{\beta+2}e^{\lambda x_i}+\sum_{i=1}^{J}R_ix_i^{\beta+2}e^{\lambda x_i}+R_nx_n^{\beta+2}e^{\lambda x_n}\Big),
\end{aligned}$$

σ_{ij}, $i,j=1,2,3$ are obtained by using ℓ_{ij}, $i,j=1,2,3$ and

$$\begin{aligned}
\ell_{111} &= \frac{2n}{\alpha^3}, \qquad \ell_{112}=0, \qquad \ell_{113}=0,\\
\ell_{122} &= -\sum_{i=1}^{n}x_i^{\beta}\log^2(x_i)e^{\lambda x_i}-\sum_{i=1}^{J}R_ix_i^{\beta}\log^2(x_i)e^{\lambda x_i}-R_nx_n^{\beta}\log^2(x_n)e^{\lambda x_n},\\
\ell_{123} &= -\sum_{i=1}^{n}x_i^{\beta+1}\log(x_i)e^{\lambda x_i}-\sum_{i=1}^{J}R_ix_i^{\beta+1}\log(x_i)e^{\lambda x_i}-R_nx_n^{\beta+1}\log(x_n)e^{\lambda x_n},
\end{aligned}$$

$$\begin{aligned}
\ell_{133} &= -\sum_{i=1}^{n} x_i^{\beta+2} e^{\lambda x_i} - \sum_{i=1}^{J} R_i x_i^{\beta+2} e^{\lambda x_i} - R_n x_n^{\beta+2} e^{\lambda x_n}, \\
\ell_{222} &= \sum_{i=1}^{n} \frac{2}{(\beta+\lambda x_i)^3} - \alpha\big(\sum_{i=1}^{n} x_i^{\beta} \log^3(x_i) e^{\lambda x_i} \\
&\quad + \sum_{i=1}^{J} R_i x_i^{\beta} \log^3(x_i) e^{\lambda x_i} + R_n x_n^{\beta} \log^3(x_n) e^{\lambda x_n}\big) \\
\ell_{223} &= \sum_{i=1}^{n} \frac{2x_i}{(\beta+\lambda x_i)^3} - \alpha\big(\sum_{i=1}^{n} x_i^{\beta+1} \log^2(x_i) e^{\lambda x_i} \\
&\quad + \sum_{i=1}^{J} R_i x_i^{\beta+1} \log^2(x_i) e^{\lambda x_i} + R_n x_n^{\beta+1} \log^2(x_n) e^{\lambda x_n}\big), \\
\ell_{233} &= \sum_{i=1}^{n} \frac{2x_i^2}{(\beta+\lambda x_i)^3} - \alpha\big(\sum_{i=1}^{n} x_i^{\beta+2} \log(x_i) e^{\lambda x_i} \\
&\quad + \sum_{i=1}^{J} R_i x_i^{\beta+2} \log(x_i) e^{\lambda x_i} + R_n x_n^{\beta+2} \log(x_n) e^{\lambda x_n}\big), \\
\ell_{333} &= 2\sum_{i=1}^{n} \big(\frac{x_i}{\beta+\lambda x_i}\big)^3 - \alpha\big(\sum_{i=1}^{n} x_i^{\beta+3} e^{\lambda x_i} + \sum_{i=1}^{J} R_i x_i^{\beta+3} e^{\lambda x_i} + R_n x_n^{\beta+3} e^{\lambda x_n}\big).
\end{aligned}$$

To obtain the Bayes approximations of α, β and λ, by assuming $U(\theta)=\alpha$, $U(\theta)=\beta$ and $U(\theta)=\lambda$, respectively, we have

$$\begin{aligned}
\widehat{\alpha}^{Lin} &= \alpha + d_1 + \frac{1}{2}\big(A\sigma_{11} + B\sigma_{12} + C\sigma_{13}\big), \\
\widehat{\beta}^{Lin} &= \gamma + d_2 + \frac{1}{2}\big(A\sigma_{12} + B\sigma_{22} + C\sigma_{23}\big), \\
\widehat{\lambda}^{Lin} &= \theta + d_3 + \frac{1}{2}\big(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\big).
\end{aligned} \tag{16}$$

It is notable that all parameters are evaluated at $(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda})$. By applying the Lindley's approximation, we cannot obtain any credible intervals. Therefore, we propose a different method i.e. MCMC algorithm, to derive the Bayes estimation and construct the association credible intervals.

3.2 MCMC algorithm

Using the equation (12), the posterior pdfs of of α, β and λ can be obtained as

$$\alpha|\beta,\lambda,\underset{\sim}{x} \sim \Gamma\big(n+a_1, b_1 + \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} + \sum_{i=1}^{J} R_i x_i^{\beta} e^{\lambda x_i} + R_n x_n^{\beta} e^{\lambda x_n}\big),$$

$$\pi(\beta|\alpha,\lambda,\underset{\sim}{x}) \propto \beta^{a_2-1}\big(\prod_{i=1}^{n}(\beta+\lambda x_i)\big) \times \big(\prod_{i=1}^{n} x_i^{\beta-1}\big)$$

$$\times e^{-\alpha\left(\sum\limits_{i=1}^{n} x_i^{\beta} e^{\lambda x_i}+\sum\limits_{i=1}^{J} R_i x_i^{\beta} e^{\lambda x_i}+R_n x_n^{\beta} e^{\lambda x_n}\right)-b_2\beta},$$

$$\pi(\lambda|\alpha,\beta,\underset{\sim}{x}) \propto \lambda^{a_3-1}\Big(\prod_{i=1}^{n}(\beta+\lambda x_i)\Big) \times e^{\lambda\sum\limits_{i=1}^{n} x_i-\alpha\left(\sum\limits_{i=1}^{n} x_i^{\beta} e^{\lambda x_i}+\sum\limits_{i=1}^{J} R_i x_i^{\beta} e^{\lambda x_i}+R_n x_n^{\beta} e^{\lambda x_n}\right)-b_3\lambda}.$$

Generating samples from the posterior pdf of α is very simple. Because the posterior pdfs of β and λ are not a well known distribution, we use the Metropolis-Hastings method with normal proposal distribution to generate samples from them. Therefore the MCMC algorithm is as
1. Start with initial value $(\alpha_0,\ \beta_0,\ \lambda_0)$.
2. Set $t = 1$.
3. Generate α_t from

$$\Gamma\big(n+a_1, b_1+\sum_{i=1}^{n} x_i^{\beta_{t-1}} e^{\lambda_{t-1}x_i}+\sum_{i=1}^{J} R_i x_i^{\beta_{t-1}} e^{\lambda_{t-1}x_i}+R_n x_n^{\beta_{t-1}} e^{\lambda_{t-1}x_n}\big).$$

4. Generate β_t from $\pi(\beta|\alpha_{t-1},\lambda_{t-1},\underset{\sim}{x})$, using Metropolis-Hastings method.
5. Generate λ_t from $\pi(\lambda|\alpha_{t-1},\beta_{t-1},\underset{\sim}{x})$, using Metropolis-Hastings method.
6. Set $t = t + 1$.
7. Repeat steps 3-6, T_b times.

Using this algorithm, under the squared error loss function, the Bayes estimates of α, β and λ are given by

$$\widehat{\alpha}^{MC}=\frac{1}{T_b}\sum_{t=1}^{T_b}\alpha_t,\quad \widehat{\beta}^{MC}=\frac{1}{T_b}\sum_{t=1}^{T_b}\beta_t,\quad \widehat{\lambda}^{MC}=\frac{1}{T_b}\sum_{t=1}^{T_b}\lambda_t. \tag{17}$$

Furthermore, by applying the method of Chen and Shao (1999), the $100(1-\eta)\%$ HPD credible interval of α, β and λ can be constructed as follows. Order $\alpha_1,\cdots,\alpha_T$ as $\alpha_{(1)}<\cdots<\alpha_{(T)}$ and construct all the $100(1-\eta)\%$ confidence intervals of α, as

$$\big(\alpha_{(1)},\alpha_{([T(1-\eta)])}\big),\cdots,\big(\alpha_{([T\eta])},\alpha_{(T)}\big),$$

where $[T]$ symbolizes the largest integer less than or equal to T. The HPD credible interval of α is the shortest length interval. Similarly, we can construct a $100(1-\eta)\%$ HPD credible interval of β and λ.

4 Simulation study and data analysis

4.1 Simulation study

In this section, we conduct a simulation study to compare the performance of different estimates and confidence intervals. The various estimates such as MLE, AMLE and Bayes estimates, are compared in terms of mean squared errors (MSE). Also, the various intervals such as asymptotic, bootstrap and HPD intervals, are compared in terms of average lengths and coverage percentages. The simulation results has been earned by considering different values of (N, n, T), and by choosing $(\alpha, \beta, \lambda) = (0.5, 0.5, 0.5)$ in all cases. The censoring schemes used in this simulation is:

$$\begin{aligned}
(N, n) &= (20, 10),\ \text{Scheme 1} : (1^{*10}),\ \text{Scheme 2} : (2^{*5}, 0^{*5}),\ \text{Scheme 3} : (0^{*5}, 2^{*5}),\\
(N, n) &= (30, 10),\ \text{Scheme 4} : (2^{*10}),\ \text{Scheme 5} : (4^{*5}, 0^{*5}),\ \text{Scheme 6} : (0^{*5}, 4^{*5}),\\
(N, n) &= (30, 20),\ \text{Scheme 7} : (1^{*10}, 0^{*10}),\ \text{Scheme 8} : (0^{*10}, 1^{*10}),\\
&\qquad \text{Scheme 9} : (2^{*5}, 0^{*15}),\\
(N, n) &= (40, 20),\ \text{Scheme 10} : (1^{*10}, 0^{*10}),\ \text{Scheme 11} : (0^{*5}, 1^{*10}, 0^{*5}),\\
&\qquad \text{Scheme 12} : (0^{*10}, 1^{*10}).
\end{aligned}$$

In simulation experiment, we employ the bootstrap intervals based on 350 re-sampling. Also, the Bayes estimates and HPD credible intervals are obtained by considering three priors as: Prior 1: $a_j = b_j = 0,\ j = 1, 2, 3$, Prior 2: $a_j = b_j = 1,\ j = 1, 2, 3$ and Prior 3: $a_j = b_j = 2,\ j = 1, 2, 3$. For the selected options of (N, n, T), different censoring schemes and different priors, the average estimates and MSE of the MLE, AMLE, Lindley and MCMC Bayes estimates of the unknown parameters over 1000 replications are achieved by the equations (5), (8), (16), (17), respectively. We noted that AMLE's of α and β are obtained when $\lambda = 0$. The results are reported in Tables 1 and 2. Moreover, we provide the 95% asymptotic intervals from (9), Boot-p intervals from (10), Boot-t intervals from (11) and HPD intervals for the unknown parameters. The results are given in Tables 3 and 4.

From Tables 1 and 2, we observed that the average values and MSEs of the different estimates are close together. AMLEs have the worst performance and Bayes estimates have the best performance. Also, comparing the Bayes estimates, it is observed that the estimates based on prior 3 have the best performance and prior 2 performs as the second best estimates. Moreover, the performance of Bayes estimates which obtained by MCMC algorithm are generally better than those obtained by Lindley's approximation. As a general result, we see that when n increases, for all cases, the MSEs of the MLEs, AMLEs and Bayes estimates decrease. This can be due to the fact that as n increases, some additional information is gathered. From Tables 3 and 4, it is observed that the HPD and bootstrap intervals have the smallest and largest average lengths, respectively and the asymptotic intervals are the second best intervals. Also, we see that the Boot-p intervals perform better than Boot-t intervals. Also, comparing the HPD intervals, it is observed that prior 3 provide the best performance and it is evident that, in most cases, the HPD intervals provide the most coverage percentages. As a general result, we see that when n increases, for all cases, the average lengths decrease and the corresponding coverage percentages increase.

Table 1: The average values and MSEs for the estimates of $(\alpha, \beta, \lambda) = (0.5, 0.5, 0.5)$ under different censoring schemes and $T = 1$.

C.S		MLE		AMLE		Lindley						MCMC					
						Prior 1		Prior 2		Prior 3		Prior 1		Prior 2		Prior 3	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
1	α	0.504	0.103	0.529	0.103	0.524	0.082	0.485	0.079	0.501	0.071	0.484	0.049	0.495	0.043	0.504	0.040
	β	0.522	0.103	0.485	0.104	0.499	0.082	0.482	0.078	0.522	0.071	0.488	0.048	0.498	0.045	0.530	0.039
	λ	0.482	0.104		-	0.521	0.035	0.496	0.079	0.520	0072	0.508	0.048	0.483	0.042	0.519	0.039
2	α	0.482	0.103	0.497	0.106	0.49	0.082	0.503	0.077	0.481	0.071	0.516	0.048	0.513	0.045	0.528	0.037
	β	0.523	0.102	0.506	0.1051	0.527	0.083	0.524	0.079	0.492	0.072	0.506	0.047	0.497	0.043	0.480	0.040
	λ	0.496	0.102		-	0.481	0.082	0.524	0.078	0.493	0.072	0.497	0.048	0.490	0.043	0.520	0.039
3	α	0.489	0.104	0.504	0.104	0.484	0.085	0.488	0.079	0.490	0.071	0.517	0.050	0.516	0.044	0.497	0.037
	β	0.514	0.100	0.503	0.103	0.502	0.086	0.503	0.077	0503	0.070	0.498	0.048	0.503	0.043	0.483	0.036
	λ	0.519	0.102		-	0.514	0.084	0.485	0.076	0.487	0.071	0.529	0.047	0.525	0.044	0.515	0.037
4	α	0.482	0.102	0.507	0.102	0.494	0.084	0.497	0.079	0.489	0.073	0.527	0.048	0.494	0.041	0.517	0.038
	β	0.513	0.103	0.507	0.103	0.504	0.084	0.500	0.076	0.518	0.072	0.495	0.047	0.499	0.043	0.522	0.036
	λ	0.528	0.101		-	0.499	0.023	0.506	0.078	0.494	0.072	0.486	0.050	0.507	0.402	0.527	0.037
5	α	0.515	0.100	0.517	0.101	0.510	0.084	0.516	0.077	0.482	0.073	0.527	0.047	0.517	0.041	0.490	0.040
	β	0.514	0.101	0.496	0.109	0.526	0.082	0.523	0.078	0.486	0.072	0.489	0.047	0.520	0.040	0.519	0.034
	λ	0.516	0.101		-	0.499	0.084	0.527	0.079	0.491	0.072	0.496	0.048	0.524	0.043	0.527	0.037
6	α	0.511	0.107	0.520	0.108	0.521	0.084	0.514	0.078	0.507	0.072	0.506	0.050	0.516	0.042	0.501	0.038
	β	0.498	0.102	0.516	0.103	0.520	0.083	0.490	0.079	0.509	0.070	0.487	0.050	0.509	0.042	0.501	0.038
	λ	0.492	0.103		-	0.506	0.083	0.485	0.078	0.521	0.071	0.519	0.048	0.482	0.042	0.517	0.038
7	α	0.517	0.042	0.497	0.048	0.521	0.039	0.481	0.034	0.482	0.028	0.487	0.031	0.483	0.022	0.504	0.020
	β	0.500	0.044	0.507	0.049	0.524	0.039	0.497	0.034	0.501	0.028	0.515	0.031	0.487	0.024	0.505	0.018
	λ	0.527	0.042		-	0.529	0.038	0.501	0.033	0.488	0.089	0.509	0.030	0.529	0.023	0.501	0.020
8	α	0.498	0.043	0.481	0.049	0.537	0.038	0.508	0.032	0.522	0.030	0.487	0.031	0.513	0.022	0.489	0.020
	β	0.529	0.041	0.524	0.048	0.514	0.038	0.513	0.032	0.491	0.029	0.483	0.031	0.506	0.021	0.485	0.019
	λ	0.512	0.042		-	0.520	0.039	0.501	0.033	0.482	0.030	0.512	0.030	0.499	0.022	0.487	0.180
9	α	0.509	0.045	0.504	0.049	0.485	0.036	0.522	0.034	0.512	0.029	0.502	0.030	0.498	0.023	0.507	0.020
	β	0.506	0.042	0.488	0.046	0.484	0.038	0.490	0.032	0.525	0.028	0.502	0.030	0.481	0.025	0.512	0.019
	λ	0.486	0.043		-	0.529	0.039	0.513	0.032	0.483	0.028	0.501	0.030	0.525	0.023	0.481	0.019
10	α	0.494	0.045	0.488	0.047	0.509	0.038	0.519	0.034	0.517	0.026	0.521	0.030	0.517	0.021	0.495	0.019
	β	0.508	0.042	0.516	0.045	0.489	0.039	0.511	0.035	0.514	0.027	0.515	0.031	0.482	0.022	0.519	0.018
	λ	0.508	0.044		-	0.494	0.039	0.529	0.033	0.517	0.029	0.514	0.030	0.523	0.022	0.486	0.020
11	α	0.521	0.043	0.484	0.047	0.488	0.039	0.505	0.033	0.497	0.027	0.485	0.031	0.512	0.021	0.521	0.017
	β	0.525	0.041	0.048	0.049	0.493	0.037	0.497	0.035	0.486	0.026	0.484	0.030	0.527	0.021	0.527	0.019
	λ	0.494	0.042		-	0.518	0.039	0.503	0.033	0.507	0.028	0.524	0.031	0.490	0.022	0.522	0.019
12	α	0.522	0.041	0.495	0.048	0.530	0.039	0.502	0.032	0.505	0.028	0.490	0.030	0.513	0.024	0.516	0.017
	β	0.483	0.044	0.516	0.049	0.493	0.039	0.502	0.032	0.500	0.028	0.488	0.030	0.505	0.023	0.510	0.016
	λ	0.530	0.043		-	0.482	0.038	0.499	0.032	0.515	0.029	0.518	0.031	0.518	0.022	0.498	0.018

Table 2: The average values and MSEs for the estimates of $(\alpha, \beta, \lambda) = (0.5, 0.5, 0.5)$ under different censoring schemes and $T = 2$.

C.S		MLE		AMLE		Lindley						MCMC					
						Prior 1		Prior 2		Prior 3		Prior 1		Prior 2		Prior 3	
		Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE	Est.	MSE
1	α	0.481	0.101	0.492	0.105	0.497	0.084	0.520	0.079	0.491	0.071	0.502	0.048	0.480	0.041	0.525	0.037
	β	0.519	0.103	0.483	0.104	0.484	0.084	0.501	0.077	0.501	0.071	0.503	0.048	0.485	0.045	0.493	0.038
	λ	0.485	0.102	-		0.494	0.083	0.525	0.078	0.518	0.072	0.482	0.048	0.491	0.042	0.487	0.040
2	α	0.484	0.103	0.510	0.103	0.517	0.081	0.521	0.080	0.510	0.072	0.485	0.050	0.519	0.043	0.494	0.040
	β	0.517	0.102	0.508	0.103	0.528	0.081	0.527	0.080	0.509	0.071	0.501	0.049	0.514	0.040	0.512	0.040
	λ	0.495	0.102	-		0.521	0.081	0.491	0.079	0.508	0.073	0.513	0.049	0.508	0.043	0.486	0.039
3	α	0.501	0.103	0.487	0.108	0.505	0.084	0.508	0.075	0.493	0.074	0.495	0.047	0.515	0.045	0.526	0.037
	β	0.485	0.102	0.518	0.104	0.514	0.085	0.492	0.076	0.483	0.074	0.486	0.046	0.493	0.043	0.504	0.036
	λ	0.497	0.103	-		0.508	0.083	0.516	0.078	0.495	0.073	0.515	0.049	0.482	0.043	0.515	0.038
4	α	0.513	0.104	0.495	0.109	0.489	0.082	0.507	0.079	0.516	0.072	0.515	0.048	0.528	0.044	0.496	0.037
	β	0.529	0.101	0.524	0.103	0.501	0.083	0.515	0.079	0.487	0.072	0.488	0.047	0.511	0.041	0.517	0.036
	λ	0.523	0.102	-		0.499	0.083	0.522	0.078	0.518	0.073	0.499	0.050	0.516	0.041	0.524	0.038
5	α	0.529	0.103	0.514	0.106	0.526	0.083	0.500	0.077	0.504	0.072	0.509	0.048	0.482	0.043	0.495	0.037
	β	0.523	0.104	0.499	0.105	0.517	0.082	0.491	0.076	0.490	0.071	0.503	0.050	0.527	0.044	0.492	0.040
	λ	0.515	0.104	-		0.530	0.082	0.484	0.077	0.521	0.073	0.503	0.050	0.483	0.042	0.490	0.040
6	α	0.518	0.101	0.509	0.101	0.520	0.084	0.527	0.078	0.514	0.073	0.503	0.046	0.487	0.044	0.481	0.037
	β	0.520	0.103	0.530	0.104	0.487	0.084	0.513	0.078	0.514	0.072	0.509	0.050	0.503	0.045	0.520	0.038
	λ	0.515	0.102	-		0.495	0.082	0.490	0.080	0.516	0.072	0.501	0.050	0.494	0.042	0.482	0.038
7	α	0.486	0.044	0.490	0.047	0.501	0.039	0.499	0.034	0.481	0.028	0.500	0.031	0.517	0.020	0.521	0.020
	β	0.522	0.041	0.501	0.048	0.504	0.038	0.506	0.032	0.525	0.026	0.518	0.030	0.504	0.022	0.522	0.020
	λ	0.521	0.045	-		0.504	0.040	0.495	0.032	0.497	0.028	0.504	0.305	0.500	0.023	0.489	0.019
8	α	0.510	0.043	0.517	0.046	0.524	0.038	0.491	0.033	0.511	0.027	0.513	0.030	0.484	0.024	0.504	0.019
	β	0.524	0.043	0.515	0.046	0.520	0.036	0.518	0.034	0.514	0.028	0.524	0.030	0.503	0.023	0.495	0.018
	λ	0.491	0.044	-		0.520	0.366	0.493	0.035	0.483	0.030	0.501	0.300	0.482	0.024	0.490	0.019
9	α	0.488	0.043	0.530	0.046	0.516	0.037	0.493	0.034	0.488	0.026	0.497	0.030	0.503	0.023	0.509	0.019
	β	0.487	0.043	0.501	0.048	0.524	0.038	0.528	0.032	0.521	0.028	0.488	0.031	0.515	0.022	0.501	0.020
	λ	0.509	0.042	-		0.484	0.038	0.493	0.031	0.520	0.029	0.490	0.302	0.489	0.021	0.510	0.020
10	α	0.497	0.045	0.482	0.046	0.525	0.040	0.489	0.035	0.516	0.027	0.520	0.031	0.510	0.023	0.527	0.019
	β	0.503	0.045	0.483	0.049	0.498	0.037	0.487	0.035	0.504	0.029	0.518	0.030	0.517	0.024	0.499	0.018
	λ	0.526	0.042	-		0.483	0.039	0.529	0.034	0.516	0.030	0.507	0.030	0.504	0.024	0.499	0.019
11	α	0.518	0.044	0.517	0.047	0.504	0.035	0.507	0.034	0.505	0.029	0.481	0.031	0.487	0.025	0.507	0.020
	β	0.486	0.042	0.520	0.048	0.504	0.038	0.521	0.033	0.482	0.027	0.484	0.030	0.528	0.025	0.523	0.019
	λ	0.500	0.045	-		0.515	0.040	0.509	0.033	0.494	0.029	0.481	0.030	0.530	0.022	0.502	0.020
12	α	0.481	0.042	0.495	0.046	0.493	0.035	0.485	0.033	0.510	0.028	0.517	0.030	0.491	0.022	0.523	0.020
	β	0.505	0.045	0.487	0.048	0.503	0.037	0.520	0.035	0.502	0.028	0.484	0.030	0.491	0.023	0.509	0.019
	λ	0.527	0.043	-		0.497	0.038	0.505	0.032	0.507	0.028	0.502	0.031	0.526	0.024	0.498	0.019

Table 3: Average length and coverage percentage for the estimates of $(\alpha,\beta) = (0.5, 0.5)$ under different censoring scheme and $T = 1$.

C.S		Asymp.		Boot-t		Boot-p		HPD					
								Prior 1		Prior 2		Prior 3	
		Length	C.P	Length	C.P	Length	C.P	Length	C.P	Length	C.P	Length	C.P
1	α	0.476	0.902	0.553	0.871	0.500	0.897	0.406	0.912	0.365	0.916	0.335	0.932
	β	0.477	0.903	0.556	0.872	0.503	0.899	0.408	0.913	0.366	0.917	0.333	0.933
	λ	0.476	0.900	0.560	0.870	0.507	0.898	0.409	0.910	0.361	0.918	0.337	0.930
2	α	0.477	0.901	0.558	0.871	0.501	0.897	0.406	0.912	0.361	0.919	0.337	0.932
	β	0.471	0.902	0.554	0.872	0.502	0.897	0.403	0.914	0.360	0.919	0.336	0.932
	λ	0.474	0.903	0.554	0.873	0.509	0.899	0.407	0.912	0.363	0.919	0.331	0.930
3	α	0.480	0.902	0.558	0.872	0.505	0.897	0.405	0.913	0.363	0.916	0.337	0.931
	β	0.471	0.903	0.556	0.870	0.506	0.896	0.408	0.912	0.367	0.918	0.335	0.932
	λ	0.472	0.903	0.555	0.873	0.509	0.899	0.409	0.911	0.367	0.918	0.330	0.933
4	α	0.476	0.902	0.557	0.871	0.505	0.898	0.404	0.911	0.360	0.919	0.330	0.932
	β	0.476	0.903	0.558	0.870	0.504	0.899	0.406	0.912	0.361	0.918	0.336	0.932
	λ	0.474	0.902	0.554	0.872	0.508	0.897	0.406	0.912	0.369	0.919	0.331	0.933
5	α	0.480	0.903	0.558	0.872	0.502	0.899	0.406	0.912	0.370	0.918	0.339	0.932
	β	0.476	0.902	0.551	0.872	0.508	0.897	0.405	0.913	0.368	0.918	0.330	0.930
	λ	0.479	0.903	0.533	0.872	0.510	0.897	0.408	0.910	0.367	0.918	0.331	0.932
6	α	0.475	0.903	0.556	0.871	0.505	0.899	0.402	0.912	0.364	0.918	0.339	0.933
	β	0.478	0.902	0.553	0.872	0.504	0.898	0.401	0.913	0.362	0.919	0.338	0.932
	λ	0.471	0.901	0.551	0.871	0.502	0.899	0.406	0.912	0.362	0.919	0.331	0.932
7	α	0.303	0.930	0.351	0.922	0.340	0.929	0.276	0.938	0.252	0.940	0.214	0.949
	β	0.300	0.930	0.355	0.923	0.331	0.929	0.270	0.938	0.250	0.942	0.215	0.950
	λ	0.304	0.931	0.350	0.922	0.332	0.928	0.275	0.939	0.258	0.942	0.212	0.950
8	α	0.303	0.931	0.353	0.923	0.339	0.929	0.277	0.939	0.254	0.942	0.213	0.948
	β	0.304	0.930	0.359	0.920	0.332	0.929	0.277	0.938	0.253	0.942	0.211	0.949
	λ	0.301	0.931	0.359	0.920	0.337	0.928	0.275	0.938	0.253	0.940	0.220	0.950
9	α	0.302	0.930	0.355	0.922	0.337	0.929	0.273	0.938	0.251	0.943	0.219	0.949
	β	0.307	0.930	0.352	0.922	0.337	0.928	0.275	0.938	0.254	0.942	0.216	0.948
	λ	0.303	0.931	0.356	0.923	0.339	0.929	0.275	0.939	0.259	0.940	0.211	0.949
10	α	0.309	0.931	0.358	0.920	0.338	0.928	0.276	0.938	0.257	0.942	0.216	0.950
	β	0.303	0.931	0.359	0.922	0.337	0.929	0.276	0.938	0.256	0.942	0.212	0.950
	λ	0.308	0.930	0.352	0.923	0.337	0.928	0.277	0.938	0.253	0.943	0.219	0.949
11	α	0.309	0.930	0.357	0.922	0.332	0.929	0.276	0.939	0.252	0.940	0.210	0.948
	β	0.302	0.931	0.354	0.922	0.335	0.929	0.276	0.939	0.253	0.941	0.217	0.949
	λ	0.303	0.930	0.359	0.923	0.335	0.928	0.277	0.938	0.250	0.942	0.211	0.950
12	α	0.301	0.931	0.352	0.923	0.332	0.929	0.275	0.939	0.252	0.940	0.217	0.950
	β	0.302	0.931	0.353	0.920	0.332	0.928	0.279	0.938	0.251	0.940	0.216	0.949
	λ	0.380	0.930	0.353	0.921	0.332	0.927	0.279	0.939	0.253	0.941	0.219	0.949

Table 4: Average length and coverage percentage for the estimates of $(\alpha, \beta) = (0.5, 0.5)$ under different censoring scheme and $T = 2$.

C.S		Asymp.		Boot-t		Boot-p		HPD					
								Prior 1		Prior 2		Prior 3	
		Length	C.P	Length	C.P	Length	C.P	Length	C.P	Length	C.P	Length	C.P
1	α	0.473	0.902	0.555	0.872	0.500	0.897	0.405	0.912	0.370	0.918	0.332	0.930
	β	0.472	0.902	0.558	0.872	0.502	0.899	0.408	0.911	0.367	0.919	0.334	0.931
	λ	0.471	0.900	0.558	0.870	0.503	0.898	0.406	0.911	0.361	0.919	0.330	0.931
2	α	0.473	0.901	0.559	0.871	0.506	0.899	0.407	0.912	0.364	0.918	0.333	0.930
	β	0.476	0.902	0.560	0.871	0.502	0.897	0.406	0.912	0.367	0.916	0.335	0.932
	λ	0.478	0.902	0.553	0.871	0.501	0.897	0.405	0.912	0.362	0.919	0.337	0.932
3	α	0.477	0.901	0.551	0.872	0.506	0.899	0.406	0.912	0.370	0.918	0.333	0.931
	β	0.470	0.903	0.556	0.873	0.504	0.898	0.403	0.913	0.363	0.917	0.339	0.930
	λ	0.472	0.902	0.559	0.872	0.503	0.899	0.405	0.910	0.370	0.917	0.335	0.931
4	α	0.473	0.903	0.552	0.872	0.501	0.897	0.404	0.912	0.364	0.917	0.337	0.932
	β	0.475	0.902	0.551	0.871	0.509	0.899	0.408	0.913	0.367	0.918	0.333	0.931
	λ	0.479	0.903	0.552	0.872	0.503	0.897	0.406	0.910	0.366	0.919	0.334	0.932
5	α	0.473	0.902	0.559	0.872	0.504	0.899	0.405	0.912	0.362	0.918	0.332	0.930
	β	0.475	0.901	0.552	0.870	0.502	0.899	0.406	0.913	0.363	0.919	0.332	0.931
	λ	0.472	0.903	0.560	0.872	0.507	0.897	0.405	0.912	0.364	0.918	0.340	0.932
6	α	0.477	0.902	0.552	0.872	0.503	0.898	0.402	0.913	0.364	0.919	0.331	0.930
	β	0.472	0.903	0.551	0.870	0.503	0.899	0.407	0.910	0.369	0.917	0.335	0.931
	λ	0.471	0.902	0.552	0.871	0.502	0.899	0.409	0.911	0.362	0.918	0.335	0.932
7	α	0.305	0.930	0.352	0.922	0.332	0.928	0.276	0.939	0.250	0.942	0.216	0.949
	β	0.302	0.931	0.360	0.923	0.340	0.928	0.277	0.938	0.256	0.943	0.212	0.950
	λ	0.303	0.931	0.352	0.920	0.339	0.929	0.279	0.938	0.257	0.940	0.219	0.950
8	α	0.310	0.930	0.357	0.920	0.339	0.928	0.271	0.939	0.257	0.942	0.214	0.949
	β	0.300	0.930	0.356	0.920	0.335	0.929	0.278	0.938	0.255	0.943	0.220	0.949
	λ	0.303	0.930	0.360	0.921	0.334	0.929	0.274	0.939	0.253	0.940	0.213	0.949
9	α	0.303	0.932	0.352	0.923	0.337	0.928	0.279	0.938	0.255	0.942	0.211	0.950
	β	0.305	0.931	0.353	0.922	0.337	0.927	0.276	0.937	0.259	0.940	0.212	0.950
	λ	0.309	0.930	0.358	0.923	0.338	0.928	0.279	0.938	0.256	0.942	0.236	0.949
10	α	0.308	0.932	0.356	0.922	0.334	0.928	0.276	0.937	0.252	0.943	0.216	0.949
	β	0.303	0.931	0.357	0.922	0.333	0.929	0.277	0.939	0.253	0.943	0.215	0.950
	λ	0.303	0.931	0.350	0.923	0.334	0.928	0.272	0.938	0.250	0.942	0.219	0.950
11	α	0.307	0.930	0.358	0.923	0.331	0.927	0.271	0.939	0.250	0.940	0.215	0.949
	β	0.303	0.930	0.360	0.923	0.330	0.927	0.274	0.938	0.251	0.941	0.213	0.949
	λ	0.305	0.931	0.357	0.920	0.333	0.928	0.273	0.937	0.252	0.941	0.213	0.950
12	α	0.304	0.932	0.353	0.920	0.335	0.928	0.274	0.938	0.254	0.942	0.219	0.949
	β	0.307	0.931	0.356	0.921	0.336	0.929	0.278	0.939	0.259	0.941	0.219	0.948
	λ	0.306	0.930	0.359	0.921	0.337	0.928	0.280	0.939	0.257	0.942	0.218	0.950

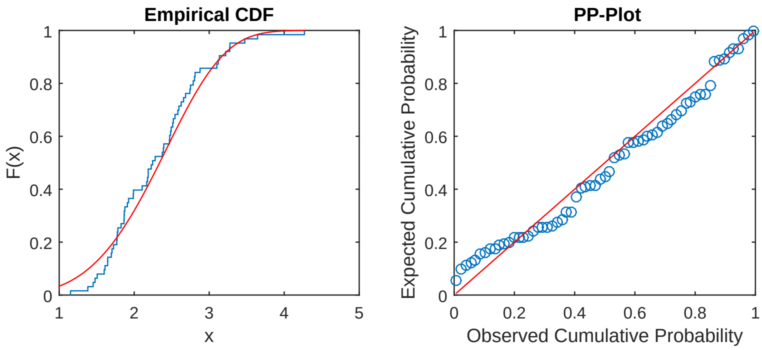

Figure 2: Empirical distribution function (left) and the PP-plot (right) for the data set.

4.2 Data Analysis

In this section, we analyze a real data set to illustrative proposes. This data which is reported by Badar and Priest (1982) describes the strengths of single carbon fibers, which measured in GPA. Single fibers were tested under tension at gauge lengths of 10mm.

To analyze the data set, we subtract 0.75 from all points of the data set. It is notable that, by this work, the statistical inference does not change. Several authors analyzed these data sets such as Kohansal and Kazemi (2019), Kohansal and Rezakhah (2019). Now, first, it is checked that the mW distribution can be used to analyze this data set. After fitting the distribution, $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\lambda}$, the Kolmogorov-Smirnov distance and the corresponding p-value are 0.0234, 2.9563, 0.3750, 0.0829, 0.7487, respectively. From the p-value, we see that the mW distribution is a suitable fit for the data set. Also, we provide the empirical distribution function and the PP-plot in Figure 2. For illustrative aims, an AT-II PC scheme has been considered as $(1^{*10}, 2^{*10}, 0^{*13})$, $T = 3.5$. For complete data set and this censoring scheme, we have obtained the values of MLE and Bayes estimation of unknown parameters with the non informative prior assumption, i.e., $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$, via two approximation methods. Also, the 95% asymptotic and HPD intervals are evaluated. We have given the results in Table 5.

Comparing the different estimates, we observe that the HPD credible intervals are slightly smaller than the asymptotic confidence intervals (CI), therefore we should apply the Bayesian inference if it is available.

Table 5: Results in real data.

		MLE	Asymp. CI	Lindley	MCMC	HPD
Complete	α	0.0234	(0.0014,0.3488)	0.0394	0.0230	(0.0094,0.3018)
	β	2.9563	(1.5214,6.8514)	3.1542	2.9422	(1.9543,6.1254)
	λ	0.3750	(0.1423,0.8220)	0.4158	0.3745	(0.1674,0.7647)
Censored	α	0.0512	(0.0015,0.4582)	0.0615	0.0510	(0.0090,0.4035)
	β	3.7548	(1.8547,8.0255)	3.9458	3.7485	(1.6222,7.5846)
	λ	0.4125	(0.1524,0.9958)	0.4325	0.4115	(0.1320,0.8985)

5 Discussion and conclusions

The estimation of the unknown parameters for a modified Weibull distribution, under the AT-II PC scheme, is studied in this paper. First we obtained the MLE's parameters. As we see in section 2.1, the MLE of parameters cannot be derived in a closed form and we should the numerical method to solve the equations. So, we earned the approximate MLE of the parameters α and β which have the closed forms, when $\lambda = 0$. Also, by obtaining the asymptotic distribution of the parameters, the asymptotic intervals of them are derived. Moreover, we achieved two bootstrap confidence intervals of the parameters. Furthermore, we infer of the Bayesian estimations. Because the Bayes estimations of the parameters have not the closed form, we use two approximate method: Lindley's approximation and MCMC algorithm. By utilizing MCMC algorithm, the HPD credible intervals of the parameters are constructed.

We compared the performance of different methods by Monte Carlo simulations. From the simulation results, we observed that the average values and MSEs of the different estimates are close together. AMLEs have the worst performance and Bayes estimates have the best performance. Also, comparing the Bayes estimates, it is observed that the estimates based informative priors have the better performance than the non-informative priors. Moreover, the performance of Bayes estimates which obtained by MCMC algorithm are generally better than those obtained by Lindley's approximation. Moreover, comparing the different confidence intervals, it is observed that the bootstrap and HPD intervals have the largest and smallest average lengths, respectively. Also, we see that the asymptotic intervals are the second best intervals and Boot-p intervals perform better than Boot-t intervals.

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