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On survival function of the generalized δ-shock model based on Polya process

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Abstract: The shock models have attracted a great deal of attention because of their important role in engineering systems. If the time between two successive shocks is less than the pre-defined threshold δ, the system fails, which is called the δ-shock model. In this article, we studied the generalized δ-shock model with two types of arrival shocks under a Polya process which has dependent interarrival times. The survival function and the mean lifetime of this system are obtained. Finally, some illustrative examples are presented.

Keywords: δ-shock model; Interarrival times; Polya process; Survival function.
Mathematics Subject Classification (2010): 62N05, 90B25, 62F03, 62F10.

1 Introduction

As a special shock model considers, the δ-shock model proposed by Li et al. (1999) and Li et al. (1999) which if the interarrival time between two shocks is shorter than a prespecified threshold δ, the system fails. This kind of shock model is useful for systems that need a period to recover from the shock. They studied the lifetime properties of the model that the shocks arrive according to a Poisson process. The δ-shock model is widely utilized in many areas such as electrical systems, inventory theory, earthquake modeling, insurance mathematics. Also, some generalizations were provided for the δ-shock model.

A mixed shock model is defined by Wang and Zhang (2007) in which the system fails when an extreme shock occurs or a δ-shock. Li and Kong (2007) studied the reliability function and some distribution properties for a δ-shock model under the

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nonhomogeneous Poisson process of shock arrivals. Li and Zhao (2007) studied a general lifetime distribution for the δ-shock model of complex systems with independent and identical components. Eryilmaz (2012) presented a generalization of the δ-shock model by using the δ-shock model based on runs and obtained the survival function and mean failure time for the system. Wang and Peng (2016) generalized the δ-shock model by assuming that the system fails when two types of shocks occur under the homogeneous Poisson process of shock arrivals which if a shock happens before the system recovered from a prior shock, the system exterminates. Cha and Finkelstein (2016) studied shock models under the generalized Polya process of shocks and derived survival and the failure rate functions based on the extreme shock model. Eryilmaz (2017) investigated the δ-shock model under a Polya process of shock arrival which has dependent interarrival times and obtained survival function and mean a lifetime of the system. Tuncel and Eryilmaz (2018) investigated the survival function and the meantime to system failure considering the proportional hazard rate model. Recently, Lorvand et al. (2019) generalized the δ-shock models to a mixed setup for the multi-state systems. Their proposed model is useful for the system which has the partially working state related to happen each interarrival time between two successive shocks in a specific critical interval.

In this article, we present a new setup of the generalized δ-shock model defined by Wang and Peng (2016) when the interarrival times are dependent and exchangeable. The continuing of the paper is organized as follows. In Section 2, some notations and the assumption of models are provided. The survival function is obtained in Section 3. In Section 4, the Mean Lifetime of the generalized δ-shock Model is investigated. The illustrative examples are presented to evaluate the results in Section 5. Finally, Concluding remarks are given in Section 6.

Notations

$N(t)$	Number of increments in Polya process
$N_i(t)$	Number of ith shocks occurred in the interval time $(0, t], i = 1, 2$
n_i	Realization of $N_i(t), i = 1, 2$
p	The Probability of a shock is type 1
q	The Probability of a shock is type 2
X_n	Interarrival time between the $(n-1)$th and nth shocks, $n = 1, 2, \ldots$
$F(t)$	Cumulative distribution function (CDF) of the interarrival time $X_n, n = 1, 2, \ldots$
δ_i	Arrival time for a type i shock, $i = 1, 2$
Z_n	Type of the nth shock, equal to 1 or 2, $n = 1, 2, \ldots$
T	Lifetime of the δ-shock model with two types of shocks

2 Model assumptions

In this section, the assumption of generalized δ-shock model is presented based on two types of shocks.

Suppose a generalized δ-shock model for a system with one component affected by two types of shocks according to the upcoming assumption.

Assumption. A newly installed system at time $t = 0$ is deal with external shocks that include two types of shocks so that shock is type 1 with probability p or is Type

2 with probability $q = 1 - p$. Therefore, the shocks are independent and based on a Polya process $\{N_t, t \geq 0\}$ enter the system.

Lemma 2.1. *Eryilmaz (2017) Polya process is a special case of mixed Poisson process with the following mass probability function*

$$\begin{aligned} P(N(t) = n) &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} dH(\lambda) \\ &= \binom{\alpha + n - 1}{n} \left(\frac{t}{t+\beta}\right)^n \left(\frac{\beta}{\beta + t}\right)^\alpha, \qquad \forall\, n = 0, 1, \ldots, \end{aligned} \tag{1}$$

where H is the Gamma distribution with density

$$dH(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha - 1} e^{-\beta\lambda}.$$

Remark 2.2. *By considering the dependent and exchangeable interarrival times X_i, $i = 1, 2, \ldots, n$, the marginal distribution of X_i is supposed Pareto with cumulative distribution function*

$$P(X_i \leq t) = 1 - \left(\frac{\beta}{\beta + t}\right)^\alpha, \qquad \forall\, t \geq 0.$$

3 Survival function of the generalized δ-shock model

In this section, the survival function of the generalized δ-shock model with two types of shocks is studied. The next lemma is required to obtain the survival function of this model.

Lemma 3.1. *Assume that $N(t), t \geq 0$ is a homogeneous Poisson process (HPP) with rate λ, and $X_1, X_2, \cdots, X_n$ denote the interarrival times of the process. Given $N(t) = n$, then*

$$P(X_2 > a, X_3 > a, \cdots, X_n > a | N(t) = n) = \left(1 - \frac{(n-1)a}{t}\right)_+^n, \tag{2}$$

for any constant $a > 0$ and $y_+ = \max(y, 0)$; Li et al. (1999), Li et al. (1999) and Eryilmaz (2012).

Once a shock happens before the system recovered from a prior shock, the system exterminates. So, it is important to obtain the survival function. The survival function of this shock model is given as follows

$$P(T > t) = \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} P(T > t, N_1(t) = n_1, N_2(t) = n_2). \tag{3}$$

To get the survival function (3), there are three cases. Firstly, no shock enters the system in $[0, t]$. So,

$$P(T > t, N_1(t) = 0, N_2(t) = 0) = \left(\frac{\beta}{\beta + t}\right)^\alpha. \tag{4}$$

Secondly, supposing that only one type of shock enters the system in $[0, t]$. If the first type of shock occurred, the survival function is

$$\begin{aligned}
&P(T > t, N_1(t) = n_1, N_2(t) = 0) \\
&\quad = P(X_1 > \delta_1, \ldots, X_{n_1} > \delta_1 | N_1(t) = n_1, N_2(t) = 0) P(N_1(t) = n_1, N_2(t) = 0) \\
&\quad = P(X_1 > \delta_1, \ldots, X_{n_1} > \delta_1 | N_1(t) = n_1, N_2(t) = 0) P(N_1(t) = n_1) P(N_2(t) = 0) \\
&\quad = p \left(\frac{\beta}{\beta + t}\right)^{2\alpha} \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \binom{\alpha + n_1 - 1}{n_1} \left(\frac{t - n_1\delta_1}{t + \beta}\right)_+^{n_1}, \qquad \forall\, n_1 = 1, 2, \cdots, \qquad (5)
\end{aligned}$$

where the last statement holds by using lemma 3.1. Similar to the second type of shock, for $n_2 = 1, 2, \cdots$, we have

$$\begin{aligned}
P(T > t, N_1(t) = 0, N_2(t) = n_2) \quad &= \quad (1 - p) \left(\frac{\beta}{\beta + t}\right)^{2\alpha} \\
&\quad \times \sum_{n_2=0}^{[\frac{t}{\delta_2}]} \binom{\alpha + n_2 - 1}{n_1} \left(\frac{t - n_2\delta_2}{t + \beta}\right)_+^{n_2}. \qquad (6)
\end{aligned}$$

Finally, for the case which both two types of shocks enter the system in $[0, t]$. By investigating the probability $P(T > t, N_1(t) = n_1, N_2(t) = n_2)$, the survival function of our proposing model can be obtained as the following theorem.

Theorem 3.2. *The survival function of the proposing generalized δ-shock model is*

$$\begin{aligned}
P(T > t) \quad = \quad & \left(\frac{\beta}{\beta + t}\right)^{\alpha} + p^{n_1} \sum_{n_1=1}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} \binom{\alpha + n_1 - 1}{n_1} \binom{\alpha + n_2 - 1}{n_2} \\
& \times \left(\frac{t'}{\beta + t'}\right)^{n} \left(\frac{\beta}{\beta + t'}\right)^{2\alpha} \left(\frac{\beta}{\beta + (n_1 - 1)\delta_1 + n_2\delta_2}\right)^{\alpha} \\
& + (1 - p)^{n_2} \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=1}^{[\frac{t}{\delta_2}]} \binom{\alpha + n_1 - 1}{n_1} \binom{\alpha + n_2 - 1}{n_2} \\
& \times \left(\frac{t_1'}{\beta + t_1'}\right)^{n} \left(\frac{\beta}{\beta + t_1'}\right)^{2\alpha} \left(\frac{\beta}{\beta + n_1\delta_1 + (n_2 - 1)\delta_2}\right)^{\alpha}, \qquad (7)
\end{aligned}$$

where $t' = [t - (n_1 - 1)\delta_1 - n_2\delta_2]_+$, $t_1' = [t - n_1\delta_1 - (n_2 - 1)\delta_2]_+$ *and* $[x]$ *denotes the integer part of* x.

Proof. By definition of our shock model, we have

$$\begin{aligned}
P(T > t) \quad &= \quad \sum_{n_1=0}^{[\frac{t}{\delta_1}]} \sum_{n_2=0}^{[\frac{t}{\delta_2}]} P(T > t, N_1(t) = n_1, N_2(t) = n_2) \\
&= \quad P(T > t, N_1(t) = 0, N_2(t) = 0)
\end{aligned}$$

$$+p^{n_1}\sum_{n_1=1}^{[\frac{t}{\delta_1}]}\sum_{n_2=0}^{[\frac{t}{\delta_2}]} P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=1)$$
$$+(1-p)^{n_2}\sum_{n_1=0}^{[\frac{t}{\delta_1}]}\sum_{n_2=1}^{[\frac{t}{\delta_2}]} P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=2) \quad (8)$$

So, we evaluate the probability which both types of shocks have happened in $[0,t]$ as follows

$$P(T>t, N_1(t)=n_1, N_2(t)=n_2) = P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=1)$$
$$+P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=2). \quad (9)$$

To obtain the right-hand side of (9)

$$P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=1) = P(X_2>\delta_{Z_1}\ldots, X_n>\delta_{Z_{n1}}, Z_n=1,$$
$$X_1+X_2+\cdots+X_n<t<X_1+X_2+\cdots+X_{n+1}, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2)$$
$$= \quad P(X_2>\delta_{Z_1},\ldots,X_n>\delta_{Z_{n1}}|Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2)$$
$$\times P(Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2)P(X_1+X_2+\cdots+X_n<t<X_1+X_2$$
$$+\cdots+X_{n+1}|X_2>\delta_{Z_1},...,X_n>\delta_{Z_{n1}}, Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2). \quad (10)$$

where $n=n_1+n_2$. Denote $X_m^{'}=X_m-\delta_{Z_m}, m=2,3,\ldots,n-1$,

$$P(X_1+X_2+\cdots+X_n<t<X_1+X_2+\cdots+X_{n+1}|X_2>\delta_{Z_1},\ldots,X_n>\delta_{Z_{n-1}},$$
$$Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2)P(X_1+X_2^{'}+\cdots+X_n^{'}<[t-(n-1)\delta_1-n_2\delta_2]_+$$
$$<X_1+X_2^{'}+\cdots+X_n^{'}+X_{n+1}|Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2)$$
$$=P(N(t^{'}=n)|Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2), \quad (11)$$

Since $X_1, X_2^{'},\ldots,X_n^{'}$ are dependent and exchangeability, the marginal distribution is Pareto and $t^{'}=[t-(n_1-1)\delta_1-n_2\delta_2]_+$. By replacing (11) into (10),

$$P(T>t, N_1(t)=n_1, N_2(t)=n_2, Z_n=1)$$
$$= \quad P(X_2>\delta_{Z_1},\ldots,X_n>\delta_{Z_{n-1}}|Z_n=1, \sum_{k=1}^{n}\phi_k=n_1, \sum_{k=1}^{n}\psi_k=n_2)$$

$$
\begin{aligned}
&\times P(N(t^{'} = n), Z_n = 1, \sum_{k=1}^{n} \phi_k = n_1, \sum_{k=1}^{n} \psi_k = n_2) \\
= \quad & \left(\frac{\beta}{\beta + (n_1 - 1)\delta_1 + n_2\delta_2}\right)_+^{\alpha} P(N(t^{'} = n), Z_n = 1, \sum_{k=1}^{n} \phi_k = n_1, \sum_{k=1}^{n} \psi_k = n_2) \\
&\times \binom{\alpha + n_1 - 1}{n_1} \binom{\alpha + n_2 - 1}{n_2} \left(\frac{t^{'}}{t' + \beta}\right)^{n_1 + n_2} \\
&\times \left(\frac{\beta}{\beta + t'}\right)^{2\alpha} \left(\frac{\beta}{\beta + (n_1 - 1)\delta_1 + n_2\delta_2}\right)_+^{\alpha}. \qquad (12)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
P(T > t, N_1(t)n_1, N_2(t) = n_2, Z_n = 2) \quad = \quad & \binom{\alpha + n_1 - 1}{n_1} \binom{\alpha + n_2 - 1}{n_2} \\
&\times \left(\frac{t^{'}}{t' + \beta}\right)^{n_1 + n_2} \left(\frac{\beta}{\beta + t'}\right)^{2\alpha} \\
&\times \left(\frac{\beta}{\beta + n_1\delta_1 + (n_2 - 1)\delta_2}\right)_+^{\alpha}. \qquad (13)
\end{aligned}
$$

So, the proof is completed. □

4 The Mean Lifetime of the generalized δ-shock Model

The survival function (7) of T for calculating of $E(T)$ is complex. So, the notion of stopping time utilizes to obtain the results.

The lifetime T of the generalized δ-shock model can be represented as

$$T = X_1 + X_2 + \cdots + X_\tau,$$

where $\tau = \min(n \geq 2 | X_n \leq \delta_{Z_{n-1}})$ denotes the number of shocks that enter the system afore failure and τ is a stopping time for $\{X_n, n \geq 1\}$. To compute $E(T)$, we require the following lemma.

Lemma 4.1. *Consider the random variables $X_n, n \geq 1$ are dependent and exchangeability distributed with Pareto distribution. So,*

$$
\begin{aligned}
P(\tau = m) \quad = \quad & P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z = z) P(Z = z) \\
= \quad & P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = 1) P(Z_m = 1) \\
& + P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = 2) P(Z_m = 2) \\
= \quad & p\left[\left(\frac{\beta}{\beta + (m-1)\delta_1}\right)^{\alpha} - \left(\frac{\beta}{\beta + m\delta_1}\right)^{\alpha} \right] \\
& + q\left[\left(\frac{\beta}{\beta + (m-1)\delta_2}\right)^{\alpha} - \left(\frac{\beta}{\beta + m\delta_2}\right)^{\alpha} \right], \qquad (14)
\end{aligned}
$$

Proof. Proof of $P(\tau = m)$ is similar to proof of $P(\tau = k)$ in Wang and Peng (2016) and Eryilmaz (2017),

$$\begin{aligned}
& P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}}, X_m \le \delta_{Z_m} | Z_m = 1) \\
= \; & P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}}, X_m \le \delta_{Z_m} | Z_m = 1, N_1(t) = n_1) \\
& \times P(Z_m = 1, N_1(t) = n_1) \\
= \; & P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}} | N_1(t) = n_1) P(N_1(t) = n_1) \\
& - P(X_1 > \delta_{Z_1}, \ldots, X_m > \delta_{Z_m} | N_1(t) = n_1) P(N_1(t) = n_1) \\
= \; & \left(\frac{\beta}{\beta + (m-1)\delta_1}\right)^{\alpha} - \left(\frac{\beta}{\beta + m\delta_1}\right)^{\alpha}.
\end{aligned}$$

For $Z_m = 2$,

$$P(X_1 > \delta_{Z_1}, \ldots, X_{m-1} > \delta_{Z_{m-1}}, X_m \le \delta_{Z_m} | Z_m = 2) = \left(\frac{\beta}{\beta + (m-1)\delta_2}\right)^{\alpha} - \left(\frac{\beta}{\beta + m\delta_2}\right)^{\alpha}$$

□

The mean lifetime can be computed in terms of $P(\tau = m)$.

Theorem 4.2. *The mean lifetime of the generalized δ-shock model is obtained as follows*

$$\begin{aligned}
E[T] \; = \; & (m-1)\min(\delta_1, \delta_2) + \frac{m}{P(\tau = m)} \frac{\beta}{\alpha - 1} \\
& \times \Bigg[p \Bigg\{ \left(\frac{\beta}{\beta + (m-1)\delta_1}\right)^{\alpha-1} - \left(\frac{\beta}{\beta + m\delta_1}\right)^{\alpha-1} \Bigg\} \\
& + q \Bigg\{ \left(\frac{\beta}{\beta + (m-1)\delta_2}\right)^{\alpha-1} - \left(\frac{\beta}{\beta + m\delta_2}\right)^{\alpha-1} \Bigg\} \Bigg] \\
& - \frac{\min(\delta_1, \delta_2)}{P(\tau = m)} \left[p \left(\frac{\beta}{\beta + m\delta_1}\right)^{\alpha} + q \left(\frac{\beta}{\beta + m\delta_2}\right)^{\alpha} \right].
\end{aligned} \tag{15}$$

Proof. By iterative expectation,

$$\begin{aligned}
E[T] = E[E[T|\tau = m]] \; & = \; \sum_{m=1}^{\infty} E[\sum_{i=1}^{m} X_i | \tau = m] P(\tau = m) \\
& = \; [(m-1)E[X_1|\tau = m] + E[X_m|\tau = m]] P(\tau = m),
\end{aligned}$$

where the random variables $X_1, X_2, \cdots$ are exchangeable, so $E[X_1|\tau = m] = \cdots = E[X_{m-1}|\tau = m]$. Therefore,

$$\begin{aligned}
E[X_1|\tau = m] \; & = \; \int_0^{\infty} P(X_1 > t | \tau = m) dt \\
& = \; c \int_0^{\infty} P(X_1 > \max(t, \delta_{Z_m}), X_2 > \delta_{z_2},
\end{aligned}$$

$$
\begin{aligned}
&\cdots, X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m})dt \\
= \;& c\Big[\int_{\min(\delta_1,\delta_2)}^{\infty} P(X_1 > \max(t, \delta_{Z_m}), X_2 > \delta_{Z_2}, \\
&\cdots, X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = z)P(Z_m = m)dt \\
&+ \int_0^{\min(\delta_1,\delta_2)} P(X_1 > t, X_2 > \delta_{Z_2}, \cdots, \\
&X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = z)P(Z_m = m)dt\Big] \\
= \;& c\Big[\int_{\min(\delta_1,\delta_2)}^{\infty} P(X_1 > t, X_2 > \delta_{Z_2}, \cdots, X_{m-1} > \delta_{Z_{m-1}}, \\
&X_m \leq \delta_{Z_m} | Z_m = z)P(Z_m = m)dt + \min(\delta_1, \delta_2)P(\tau = m)\Big] \\
= \;& c\Big[P(\tau = m)\delta_1 + P(\tau = m)\delta_2 \\
&+ \int_{\min(\delta_1,\delta_2)}^{\infty} P(X_1 > t, X_2 > \delta_{Z_2}, \cdots, \\
&X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = z)P(Z_m = m)dt\Big] \\
= \;& c\Big[P(\tau = m)\delta_1 + P(\tau = m)\delta_2 \\
&+ \int_{\delta_1}^{\infty} P(X_1 > t, X_2 > \delta_{Z_2}, \cdots, \\
&X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = 1)P(Z_m = 1)dt \\
&+ \int_{\delta_2}^{\infty} P(X_1 > t, X_2 > \delta_{Z_2}, \cdots, \\
&X_{m-1} > \delta_{Z_{m-1}}, X_m \leq \delta_{Z_m} | Z_m = 2)P(Z_m = 2)dt\Big] \\
= \;& c\Big[P(\tau = m)\delta_1 + P(\tau = m)\delta_2 \\
&+ \int_{\delta_1}^{\infty} p\left(\left(\frac{\beta}{\beta + (m-2)\delta_1 + t}\right)^{\alpha} - \left(\frac{\beta}{\beta + t + (m-1)\delta_1}\right)^{\alpha}\right) dt \\
&+ \int_{\delta_2}^{\infty} q\left(\left(\frac{\beta}{\beta + (m-2)\delta_2 + t}\right)^{\alpha} - \left(\frac{\beta}{\beta + t + (m-1)\delta_2}\right)^{\alpha}\right) dt\Big] \\
= \;& c\Big[P(\tau = m)\delta_1 + P(\tau = m)\delta_2 + p\frac{\beta^{\alpha}}{\alpha+1}[\beta + \delta_1 + (m-2)\delta_1]^{1-\alpha} \\
&- p\frac{\beta^{\alpha}}{\alpha+1}[\beta + \delta_1 + (m-1)\delta_1]^{1-\alpha} \\
&+ q\frac{\beta^{\alpha}}{\alpha+1}[\beta + \delta_2 + (m-2)\delta_2]^{1-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\quad -q\frac{\beta^{\alpha}}{\alpha+1}[\beta+\delta_2+(m-1)\delta_2]^{1-\alpha}\Big] \\
&= \frac{\beta}{(\alpha-1)P(\tau=m)}\Bigg[p\Bigg\{\left(\frac{\beta}{\beta+(m-1)\delta_1}\right)^{\alpha-1}-\left(\frac{\beta}{\beta+m\delta_1}\right)^{\alpha-1}\Bigg\} \\
&\quad +\delta_1+\delta_2+q\Bigg\{\left(\frac{\beta}{\beta+(m-1)\delta_2}\right)^{\alpha-1}-\left(\frac{\beta}{\beta+m\delta_2}\right)^{\alpha-1}\Bigg\}\Bigg],
\end{aligned}
$$

where $c=\frac{1}{P(\tau=m)}$. On the other hand,

$$
\begin{aligned}
E[X_m|\tau=m] &= \int_0^{\infty} P(X_m>t|\tau=m)dt \\
&= c\int_0^{\min(\delta_1,\delta_2)} P(X_1>\delta_{Z_1},\cdots,X_{m-1}>\delta_{Z_{m-1}},t<X_m\le\delta_{Z_m})dt \\
&= c\int_0^{\min(\delta_1,\delta_2)} P(X_1>\delta_{Z_1},\cdots, \\
&\quad X_{m-1}>\delta_{Z_{m-1}},t<X_m\le\delta_{Z_m}|Z_m=z)P(Z_m=z)dt \\
&= c\Bigg[\int_0^{\min(\delta_1,\delta_2)} P(X_1>\delta_{Z_1},\cdots, \\
&\quad X_{m-1}>\delta_{Z_{m-1}},t<X_m\le\delta_{Z_m}|Z_m=1)P(Z_m=1)dt \\
&\quad +\int_0^{\min(\delta_1,\delta_2)} P(X_1>\delta_{Z_1},\cdots,X_{m-1}>\delta_{Z_{m-1}}, \\
&\quad t<X_m\le\delta_{Z_m}|Z_m=2)P(Z_m=2)dt\Bigg] \\
&= \frac{1}{P(\tau=m)}\frac{\beta}{\alpha-1}\Bigg[p\Bigg\{\left(\frac{\beta}{\beta+(m-1)\delta_1}\right)^{\alpha-1}-\left(\frac{\beta}{\beta+m\delta_1}\right)^{\alpha-1}\Bigg\} \\
&\quad +q\Bigg\{\left(\frac{\beta}{\beta+(m-1)\delta_2}\right)^{\alpha-1}-\left(\frac{\beta}{\beta+m\delta_2}\right)^{\alpha-1}\Bigg\}\Bigg] \\
&= -\frac{\min(\delta_1,\delta_2)}{P(\tau=m)}\left[p\left(\frac{\beta}{\beta+m\delta_1}\right)^{\alpha}+q\left(\frac{\beta}{\beta+m\delta_2}\right)^{\alpha}\right].
\end{aligned}
$$

□

5 Example

In this section, the survival function is plotted for different values of parameters $\alpha>1,\beta,\delta_1,\delta_2$ and p. First case, setting $\delta_1=2,\delta_2=1,\alpha=1,\beta=0.5$ and differ the values of p, the survival function of the generalized δ-shock model based on Polya process can be calculated. From Figure 1, we can see that with increasing the values of p for any $t>0$, the survival function decreases. This is since the system fails when a type 2 shock happens before the system recovered from a type 1 shock. By increasing the value of p, the probability of occurring the shock from type 1 gains in the shock process, and the system is more easily exterminated by the shocks. In Figure 2, we plotted the expectation of lifetime for $\delta_1 = 2, \delta_2 = 1, \alpha = 2, \beta = 0.5$ and varying the values of p. Then, we conclude that for any m, the expectation of lifetime increases gradually with the increase of p.

Second case, setting $\delta_1 = 2, \delta_2 = 1, \alpha = 1, p = 0.5$ and differ the values of β, the survival function of the generalized δ-shock model based on the Polya process can be calculated. From Figure 1, we can see that with decreasing of β the survival function decreases slowly for any $t > 0$. The recovery of the system from both types of shocks takes long a more time because both types of shocks occur with the same probability. Then by increasing the value of β, the probability of occurring the shock from both types gains in the shock process. So, the system is more simply exterminated by the shocks. In Figure 2, we plotted the expectation of lifetime for $\delta_1 = 2, \delta_2 = 1, \alpha = 3, p = 0.5$ and varying the values of β. Then, we conclude that for any m, the expectation of lifetime increases slowly with the increase of β.

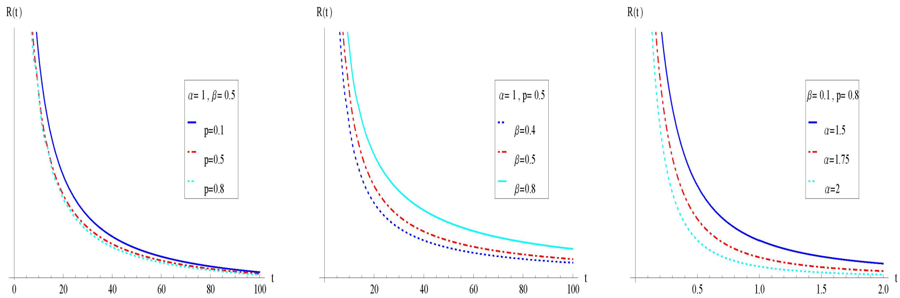

Figure 1: The survival function $R(t)$ with $\delta_1 = 2, \delta_2 = 1, \alpha = 1, \beta = 0.5$.

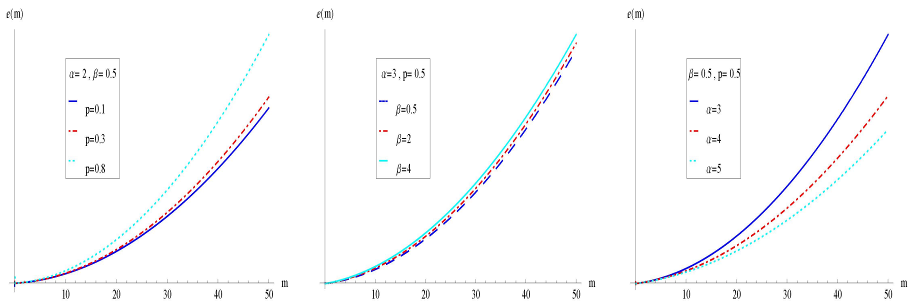

Figure 2: The expectation of life time $E[m]$ with $\delta_1 = 1, \delta_2 = 2, \alpha = 2, \beta = 0.5$, and for different p.

Third case, setting $\delta_1 = 2, \delta_2 = 1, \beta = 0.1, p = 0.8$ and modifying the values of α, the survival function of the generalized δ-shock model based on the Polya process can be calculated. From Figure 1, the survival function decreases slowly with the increase of α for any $t > 0$. The recovery of the system from both types of shocks takes long a more time because both types of shocks occur with the same probability. By increasing the value of α, the probability of occurring the shock from both types gains in the shock process. So, the system is more simply exterminated by the shocks. In Figure 2, we plotted the expectation of lifetime for $\delta_1 = 2, \delta_2 = 1, \beta = 0.5, p = 0.5$ and varying the values of α. Then, we conclude that for any m, the expectation of lifetime decreases gradually with the increase of α.

6 Conclusions

In this study, we have studied a generalized δ-shock model with two types of shocks based on the Polya process, and the recovery times δ_1 and δ_2 related to the shocks from types 1 and 2, respectively. We have derived explicit expressions for the survival function of the system. We suppose that the two types of shocks are independent, and the interarrival times between shocks are exchangeable and dependent. The illustrative examples are presented to evaluate the results.

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