Journal of Statistical Modelling: Theory and Applications
Vol. 2, No. 1, 2021, pp. 121-142
Yazd University Press 2021

JSMTA

Research Paper

Odd power generalized Weibull-G family of distributions: Model, properties and applications

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Received: December 4, 2020/ Revised: July 7, 2021 / Accepted: September 12, 2021

Abstract: A new generalized family of distributions called the odd power generalized Weibull-G family of distributions is developed. Some properties of the new family of distributions including quantile function, moments, incomplete and probability weighted moments, distribution of the order statistics and Rényi entropy are derived. Estimation of model parameters using maximum likelihood estimation technique and simulation study to examine the bias and mean square error are discussed. Applications to real data sets to illustrate the applicability of the generalized family of distributions is also given.

Keywords: Maximum likelihood estimation; Power generalized Weibull distribution.
Mathematics Subject Classification (2010): 60E05 62E15 62F30.

1 Introduction

Many statistical distributions such as Weibull, Lindley, Lomax, log-logistic, Pareto, Rayleigh distributions are widely used for fitting data in several areas such as medicine, engineering, finance, economics, and agriculture. However, in many practical situations, these statistical distributions do not provide adequate fit in modelling real life data. Thus, there is clear need for the generalizations of these distributions to gain flexibility. Statisticians proposed new families of distributions that extend the well known standard distributions by adding one or more parameters. Some of the recent known families are: generalized odd Weibull-G by Korkmaz et al. (2018), exponentiated odd log-logistic-G by Alizadeh et al. (2018), Topp Leone odd Lindley-G by Reyad et al. (2018), odd Lomax-G by Cordeiro et al. (2019), Marshall-Olkin alpha

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power-G by Nassar et al. (2019), odd inverse Pareto-G class by Aldahlan et al. (2019), the Nadarajah Haghighi Topp Leone-G family of distributions by Reyad et al. (2019), the odd exponentiated half-logistic-G family of distributions by Afify et al. (2017), a new Weibull-X family of distributions by Ahmad and Hamedan (2018), exponentiated Weibull-H by Cordeiro et al. (2017) among others.

Lai (2013) described the power generalized Weibull (PGW) distribution as one of the Weibull modifications that can give rise to non-monotonic hazard rate functions of various shapes such as a bathtub, upside-down bathtub (unimodal) or a modified bathtub. Maximum likelihood estimates (MLEs) of parameters and application of PGW distribution using Efron (1988) head-and-neck cancer clinical trial data was presented by Nikulin and Haghighi (2009). Kumar and Dey (2017) derived recurrence relations for single and product moments of order statistics from power generalized Weibull distribution. Voinov et al. (2013) constructed modified chi-squared tests based on maximum likelihood estimates (MLEs). Last but not least the generalized order statistics (GOS) of the PGW distribution was presented by Kumar and Jain (2018).

A random variable T is said to follow the power generalized Weibull distribution if its cdf and pdf are given by

$$\begin{aligned} F(t;\alpha,\beta) &= 1-\exp\left(1-[1+t^{\alpha}]^{\beta}\right), \\ f(t;\alpha,\beta) &= \alpha\beta t^{\alpha-1}(1+t^{\alpha})^{\beta-1}\exp\left(1-(1+t^{\alpha})^{\beta}\right), \end{aligned}$$

respectively, for $\alpha,\beta>0$. Alzaghal et al. (2013) defined the T-X family of distributions given by

$$F(x)=\int_a^{W(G(x))} r(t)dt.$$

Here, we consider the transformation $W(G(x;\xi))=\frac{G(x;\xi)}{1-G(x;\xi)}$ for the baseline cdf $G(x;\xi)$.

Motivation for developing this new model is the advantages presented by this extended distribution with respect to having a hazard function that exhibits both monotone and non-monotone shapes, as well as the versatility and flexibility of power generalized Weibull distributions in general, in the modeling lifetime data.

The results in this note are organized in the following manner. Section 2 contain the new OPGW-G family of distributions and its sub-families, hazard function, special cases and expansion of the density. In Section 3, quantile function, moments and generating function, probability weighted moments, the distribution of order statistics and Rényi entropy are presented. Section 4 contain the estimation of the parameters of the OPGW-G family of distributions via the method of maximum likelihood, followed by a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates in Section 5. Some applications to real data sets are given in Section 6, followed by some concluding remarks in Section 7.

2 The model, sub-families, reliabibility measures and special cases

The derivation of some of the statistical properties of the odd power generalized Weibull-G (OPGW-G) family of distributions including sub-families, hazard function, reliability measures and expansion of the density are presented in this section.

2.1 The model

The cumulative distribution function (cdf) and probability density function (pdf) of the proposed odd power generalized Weibull-G (OPGW-G) family of distributions are given by

$$\begin{aligned} F(x;\alpha,\beta,\xi) &= \int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}} \alpha\beta t^{\alpha-1}(1+t^\alpha)^{\beta-1}\exp\left(1-(1+t^\alpha)^\beta\right)dt \\ &= 1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right]^\beta\right) \qquad (1) \\ f(x;\alpha,\beta,\xi) &= \alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} \\ &\quad \times\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right]^\beta\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2}, \qquad (2) \end{aligned}$$

for $\alpha,\beta>0$ and parameter vector ξ. The hazard rate function of the OPGW-G family of distributions is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\xi) &= \frac{f(x;\alpha,\beta,\xi)}{\overline{F}(x;\alpha,\beta,\xi)} \\ &= \alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^\alpha\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} \\ &\quad \times g(x;\xi)(1-G(x;\xi))^{-2}. \end{aligned}$$

To check the identifiability of the new family, we let $\theta_1=(\alpha_1,\beta_1)$ and $\theta_2=(\alpha_2,\beta_2)$, Thus, we have

$$\begin{aligned} f_{\theta_1} &= \alpha_1\beta_1\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_1}\right]^{\beta_1-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_1-1} \\ &\quad \times\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_1}\right]^{\beta_1}\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2}, \qquad (3) \\ f_{\theta_2} &= \alpha_2\beta_2\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_2}\right]^{\beta_2-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_2-1} \\ &\quad \times\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_2}\right]^{\beta_2}\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2}. \end{aligned}$$

Then

$$f_{\theta_1} = f_{\theta_2} \Leftrightarrow a_1 - a_2 = 0, \tag{4}$$

where

$$a_1 = \alpha_1\beta_1 \left[1 + \left(\tfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_1}\right]^{\beta_1-1} \left(\tfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_1-1} \exp\left(1 - \left[1 + \left(\tfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_1}\right]^{\beta_1}\right),$$

$$a_2 = \alpha_2\beta_2 \left[1 + \left(\tfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_2}\right]^{\beta_2-1} \left(\tfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_2-1} \exp\left(1 - \left[1 + \left(\tfrac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha_2}\right]^{\beta_2}\right).$$

Thus, expression (4) is equal to zero for almost all $G(x;\xi)$ when all its coefficients are equal to zero, which is only possible when $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. Since all the parameters are restricted to be greater than zero, we conclude that the new family of distributions is identifiable: $f_{\theta_1} = f_{\theta_2} \Leftrightarrow \theta_1 = \theta_2$.

2.2 Sub-families of OPGW-G family of distributions

In this subsection, some sub-families of the OPGW-G family of distributions are presented.

- When $\beta = 1$, we obtain the Weibull-G (W-G) family of distributions (Bourguignon et al., 2004) with the cdf

$$F(x;\alpha,\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right),$$

  for $\alpha > 0$, and parameter vector ξ.

- If $\alpha = 1$, we obtain the odd Nadarajah Haghighi-G (ONH-G) family of distributions with the cdf

$$F(x;\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^{\beta}\right),$$

  for $\beta > 0$, and parameter vector ξ. This is a new family of distributions.

- If $\alpha = \beta = 1$, we obtain the odd exponential-G (OE-G) family of distributions with the cdf

$$F(x;\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right),$$

  for parameter vector ξ.

- If $\beta = 1, \alpha = 2$ we obtain the odd Rayleigh-G (OR-G) family of distributions with the cdf

$$F(x;\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{2}\right),$$

  for parameter vector ξ.

2.3 Asymptotes

In this subsection, we give some asymptotes for cdf, pdf and hazard rate functions of the OPGW-G family of distributions. To easily find asymptotes we write (1) as

$$F(x;\alpha,\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{-\alpha}\right]^{\beta}\right).$$

As $x \to 0$, we have

$$\begin{aligned}
F(x;\alpha,\beta,\xi) &= 1 - \exp\left(1 - \left[1 + \left(\frac{1}{G(x;\xi)} - 1\right)^{-\alpha}\right]^{\beta}\right) \\
&\sim 1 - \exp\left(1 - [1 + (G(x;\xi))^{\alpha}]^{\beta}\right), \\
f(x;\alpha,\beta,\xi) &\sim \alpha\beta\,[1 + (G(x;\xi))^{\alpha}]^{\beta-1}\,(G(x;\xi))^{\alpha-1}\exp\left(1 - [1 + (G(x;\xi))^{\alpha}]^{\beta}\right), \\
h(x;\alpha,\beta,\xi) &\sim \alpha\beta\,[1 + (G(x;\xi))^{\alpha}]^{\beta-1}\,(G(x;\xi))^{\alpha-1}.
\end{aligned}$$

The asymptotes for cdf, pdf and hazard rate functions of the OPGW-G family of distributions as $x \to \infty$, are given by

$$\begin{aligned}
1 - F(x;\alpha,\beta,\xi) &= \exp\left(1 - \left[1 + \left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{-\alpha}\right]^{\beta}\right) \\
&= \sum_{i=0}^{\infty} \frac{\left(1 - \left[1 + \left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{-\alpha}\right]^{\beta}\right)^{i}}{i!} \\
&\approx 1 - \left(1 - \left[1 + \left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{-\alpha}\right]^{\beta}\right).
\end{aligned}$$

Consequently, as $x \to \infty$,

$$\begin{aligned}
1 - F(x;\alpha,\beta,\xi) &\sim \left[1 + (1 - G(x;\xi))^{-\alpha}\right]^{\beta}, \\
f(x;\alpha,\beta,\xi) &\sim \frac{\alpha\beta g(x;\xi)\left[1 + (1 - G(x;\xi))^{-\alpha}\right]^{\beta-1}}{(1 - G(x;\xi))^{\alpha+1}}, \\
h(x;\alpha,\beta,\xi) &\sim \frac{\alpha\beta g(x;\xi)}{(1 - G(x;\xi))^{\alpha+1}\left[1 + (1 - G(x;\xi))^{-\alpha}\right]^{\beta}}.
\end{aligned}$$

2.4 Series expansion of density function

In this section, we present the series expansion of the OPGW-G density function. Applying the following expansions

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad (a+b)^n = \sum_{j=0}^{\infty} \binom{n}{j} a^j b^{n-j},$$
$$(1-z)^{-k} = \sum_{l=0}^{\infty} \frac{\Gamma(k+l)}{\Gamma(k)l!} z^l, \quad (1-z)^{a-1} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(a)}{\Gamma(a-k)k!} z^k,$$

for $|z| < 1$ and $k > 0$, the pdf of OPGW-G family of distributions can be written as

$$f(x;\alpha,\beta,\xi) = \sum_{w=0}^{\infty} C_{w+1} h_{w+1}(x;\xi), \tag{5}$$

where $h_{w+1}(x;\xi) = (w+1)[G(x;\xi)]^w g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $w+1 > 0$ and parameter vector ξ,

$$\begin{aligned} C_{w+1} &= \alpha\beta \sum_{i,q,j,p,l,k=0}^{\infty} \binom{\beta-1}{q} \binom{i}{j} \binom{\beta j}{p} \frac{\Gamma(\alpha(p+q+1)+1+l)}{\Gamma(\alpha(p+q+1)+1)l!} \\ &\quad \times \binom{k}{w} \frac{\Gamma(\alpha(q+p+1)+l)}{\Gamma(\alpha(q+p+1)+l-k)k!} \frac{(-1)^{j+k+w}}{i!} \frac{1}{w+1}. \end{aligned} \tag{6}$$

See the appendix for the expansions leading to (5). Consequently, the mathematical and statistical properties of the OPGW-G family of distributions follow directly from those of the exponentiated-G (E-G) distribution.

For the convergence of the series

$$\exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right) = \sum_{i=0}^{\infty} \frac{\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)^i}{i!},$$

using the ratio test, we have

$$\lim_{i\to\infty} \frac{\frac{\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)^{i+1}}{(i+1)!}}{\frac{\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)^{i}}{i!}} = \lim_{i\to\infty} \frac{\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{(i+1)} = 0.$$

Thus, the series converges for every $G(x;\xi)$.

2.5 Some special cases

In this section, we consider some special cases of the OPGW-G family of distributions, specifically when the distribution function $G(x;\xi)$ is gamma, Burr XII and power distributions, respectively.

2.5.1 OPGW-gamma distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;a,b)=\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}$ and $g(x;a,b)=\frac{x^{a-1}\exp(\frac{-x}{b})}{b^a\Gamma(a)}$ for $a,b>0$ and $x>0$. The new OPGW-gamma (OPGW-Ga) distribution has cdf and pdf given by

$$
\begin{aligned}
F(x;\alpha,\beta,a,b) &= 1-\exp\left(1-\left[1+\left(\frac{\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}{1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}\right)^{\alpha}\right]^{\beta}\right)\\
f(x;\alpha,\beta,a,b) &= \alpha\beta\left[1+\left(\frac{\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}{1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}{1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}\right)^{\alpha-1}\\
&\quad\times\exp\left(1-\left[1+\left(\frac{\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}{1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}\right)^{\alpha}\right]^{\beta}\right)\frac{x^{a-1}\exp(\frac{-x}{b})}{b^a\Gamma(a)}\\
&\quad\times\left(1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}\right)^{-2},
\end{aligned}
$$

respectively, for $\alpha,\beta,a,b>0$. The hazard rate function is given by

$$
\begin{aligned}
h_F(x;\alpha,\beta,a,b) &= \alpha\beta\left[1+\left(\frac{\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}{1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}{1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}}\right)^{\alpha-1}\\
&\quad\times\frac{x^{a-1}\exp\left(\frac{-x}{b}\right)}{b^a\Gamma(a)}\left(1-\frac{\gamma(a,\frac{x}{b})}{\Gamma(a)}\right)^{-2}.
\end{aligned}
$$

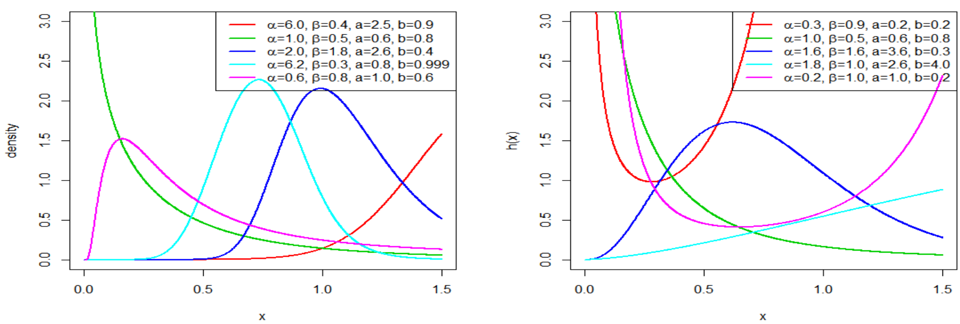

Figure 1: Density and hazard function plots for OPGW-Ga distribution

Figure 1 shows the plots of pdf and hazard functions of OPGW-Ga distribution, respectively. The pdf can take several shapes including right skewed, left skewed, almost symmetric and reverse-J shapes. The OPGW-Ga hazard function displays increasing, decreasing, bathtub and upside-down bathtub shapes.

2.5.2 OPGW-Burr XII distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x; s, c, k) = 1 - \left[1 + (\frac{x}{s})^c\right]^{-k}$ and $g(x; s, c, k) = cks^{-c}x^{c-1}\left[1 + (\frac{x}{s})^c\right]^{-k-1}$, for $s, c, k > 0$, and $x > 0$. Then the new OPGW-Burr XII (OPGW-BXII) distribution has cdf and pdf given by

$$\begin{aligned}
F(x; \alpha, \beta, s, c, k) &= 1 - \exp\left(1 - \left[1 + \left(\frac{1 - \left[1 + (\frac{x}{s})^c\right]^{-k}}{\left[1 + (\frac{x}{s})^c\right]^{-k}}\right)^{\alpha}\right]^{\beta}\right) \\
f(x; \alpha, \beta, s, c, k) &= \alpha\beta\left[1 + \left(\frac{1 - \left[1 + (\frac{x}{s})^c\right]^{-k}}{\left[1 + (\frac{x}{s})^c\right]^{-k}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1 - \left[1 + (\frac{x}{s})^c\right]^{-k}}{\left[1 + (\frac{x}{s})^c\right]^{-k}}\right)^{\alpha-1} \\
&\quad \times \exp\left(1 - \left[1 + \left(\frac{1 - \left[1 + (\frac{x}{s})^c\right]^{-k}}{\left[1 + (\frac{x}{s})^c\right]^{-k}}\right)^{\alpha}\right]^{\beta}\right) cks^{-c}x^{c-1} \\
&\quad \times \left[1 + \left(\frac{x}{s}\right)^c\right]^{-k-1}\left(\left[1 + \left(\frac{x}{s}\right)^c\right]^{-k}\right)^{-2},
\end{aligned}$$

respectively, for $\alpha, \beta, s, c, k > 0$. The hazard rate function is given by

$$\begin{aligned}
h_F(x; \alpha, \beta, s, c, k) &= \alpha\beta\left[1 + \left(\frac{1 - \left[1 + (\frac{x}{s})^c\right]^{-k}}{\left[1 + (\frac{x}{s})^c\right]^{-k}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1 - \left[1 + (\frac{x}{s})^c\right]^{-k}}{\left[1 + (\frac{x}{s})^c\right]^{-k}}\right)^{\alpha-1} \\
&\quad \times cks^{-c}x^{c-1}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-k-1}\left(\left[1 + \left(\frac{x}{s}\right)^c\right]^{-k}\right)^{-2}.
\end{aligned}$$

For $s = k = 1$, we obtain OPGW-log-logistc (OPGW-LLoG) distribution.

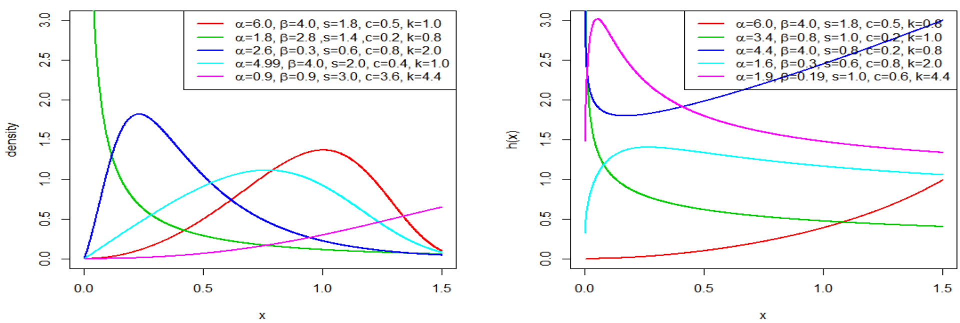

Figure 2: Density and hazard function plots for OPGW-BXII distribution

Figure 2 shows the plots of pdf and hazard functions of OPGW-Burr XII distribution, respectively. The pdf can take several shapes including right skewed, left skewed, almost symmetric and reverse-J shapes. The OPGW-Burr XII hazard function displays increasing, decreasing, bathtub and upside-down bathtub shapes.

2.5.3 OPGW-Power distribution

The cdf and pdf of the power distribution are given by $G(x;\theta,k) = (\theta x)^k$ and $g(x;\theta,k) = k\theta^k x^{k-1}$, for $\theta, k > 0$, and $x \in (0, \frac{1}{\theta})$. By replacing these in equations (1) and (2), then we obtain the new OPGW-power (OPGW-P) distribution with cdf and pdf given by

$$\begin{aligned} F(x;\alpha,\beta,\theta,k) &= 1-\exp\left(1-\left[1+\left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{\alpha}\right]^{\beta}\right) \\ f(x;\alpha,\beta,\theta,k) &= \alpha\beta\left[1+\left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{\alpha}\right]^{\beta-1}\left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{\alpha-1} \\ &\quad\times\exp\left(1-\left[1+\left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{\alpha}\right]^{\beta}\right)k\theta^k x^{k-1}\left(1-(\theta x)^k\right)^{-2}, \end{aligned}$$

respectively, for $\alpha, \beta, \theta, k > 0$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,\theta,k) &= \alpha\beta\left[1+\left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{\alpha}\right]^{\beta-1}\left(\frac{(\theta x)^k}{1-(\theta x)^k}\right)^{\alpha-1} \\ &\quad\times k\theta^k x^{k-1}\left(1-(\theta x)^k\right)^{-2}. \end{aligned}$$

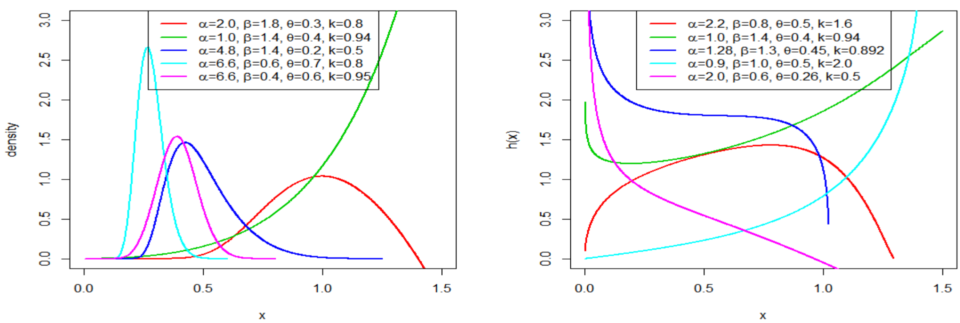

Figure 3: Density and hazard function plots for OPGW-P distribution

Figure 3 shows the plots of pdf and hazard functions of OPGW-power distribution, respectively. The pdf can take several shapes including increasing, right skewed, left skewed and almost symmetric shapes. The OPGW-power hazard function displays increasing, decreasing, bathtub and upside-down bathtub shapes.

3 Some properties of OPGW-G family

In this section, we study some main properties of OPGW-G family of distributions including quantile function, moments, order statistics and Rényi entropy.

3.1 Quantile function

The quantile function of the OPGW-G family of distributions is obtained by solving the non-linear equation:

$$F(x;\alpha,\beta,\xi)=1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)=u,$$

for $0\leq u\leq 1$, that is,

$$\frac{1}{G(x;\xi)}=\left((1-\log(1-u))^{\frac{1}{\beta}}-1\right)^{\frac{-1}{\alpha}}+1.$$

Consequently, the quantile function for the OPGW-G family of distributions is given by

$$Q_G(u;\alpha,\beta,\xi)=X=G^{-1}\left[\left(\left((1-\log(1-u))^{\frac{1}{\beta}}-1\right)^{\frac{-1}{\alpha}}+1\right)^{-1}\right], \quad (7)$$

where $Q_G(u;\alpha,\beta,\xi)$ is the quantile function of the OPGW-G family of distributions and $G^{-1}()$ is the quantile function of the baseline distribution. It follows therefore that random numbers can be generated from the OPGW-G family of distributions based on (7).

3.2 Moments and generating function

The n^{th} raw moment, μ_n' of the OPGW-G family of distributions is given by

$$\mu_n'=E(X^n)=\int_{-\infty}^{\infty}x^n f(x)dx=\sum_{w=0}^{\infty}C_{w+1}E(Y_{w+1}^n), \quad (8)$$

where $Y_{w+1}\sim Exponentiated-G(w+1,\xi)$. The moment generating function (MGF) $M_X(t)=E(e^{tX})$ is given by

$$M_X(t)=\sum_{w=0}^{\infty}C_{w+1}M_{w+1}(t),$$

where $M_{w+1}(t)$ is the mgf of Y_{w+1} and C_{w+1} is given by (6).

3.3 Probability weighted moments

The $(s,r)^{th}$ PWM of X with OPGW-G distribution denoted $K_{s,r}$ is given by

$$K_{s,r}=E(X^s(F(X))^r)=\int_{-\infty}^{\infty}x^s(F(x))^r f(x)dx.$$

Using equations (1) and (2), we can write

$$f(x)(F(x))^r \quad = \quad \alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1}g(x;\xi)$$

$$
\begin{aligned}
& \times \exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)(1-G(x;\xi))^{-2} \\
& \times \left(1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right)^{r} \\
= & \sum_{m=0}^{\infty}\binom{r}{m}(-1)^{m}\alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1} \\
& \times \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} g(x;\xi)(1-G(x;\xi))^{-2} \\
& \times \exp\left((m+1)\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right).
\end{aligned}
$$

Following the same steps of the density expansion leading to (5), we get

$$
f(x)F(x)^{r}=\sum_{w=0}^{\infty} b_{w+1}h_{w+1}(x;\xi),
$$

where

$$
\begin{aligned}
b_{w+1} = \alpha\beta & \sum_{i,q,j,p,l,k,m=0}^{\infty}\binom{r}{m}\binom{\beta-1}{q}\binom{i}{j}\binom{\beta j}{p}\frac{\Gamma(\alpha(p+q+1)+1+l)}{\Gamma(\alpha(p+q+1)+1)l!} \\
& \times \binom{k}{w}\frac{\Gamma(\alpha(q+p+1)+l)}{\Gamma(\alpha(q+p+1)+l-k)k!}\frac{(-1)^{j+k+w}}{i!}\frac{(-1)^{m}(m+1)^{i}}{w+1}.
\end{aligned}
$$

Consequently, the PWM of the OPGW-G family of distributions can be written as

$$
K_{s,r}=\int_{-\infty}^{\infty} x^{s}\sum_{w=0}^{\infty} b_{w+1}h_{w+1}(x;\xi)dx=\sum_{w=0}^{\infty} b_{w+1}\int_{-\infty}^{\infty} x^{s}h_{w+1}(x;\xi)dx.
$$

Finally, the $(s,r)^{th}$ PWM of X can be written as an infinite linear combination of the moment of the E-G distribution.

3.4 Order statistics

In this sub-section, the density function of the OPGW-G order statistics is presented as a linear combination of E-G densities.. Let $X_1, X_2, ..., X_n$ be independent and identically distributed OPGW-G random variables. Using the binomial expansion

$$
(1-F(x))^{n-t}=\sum_{m=0}^{n-t}\binom{n-t}{m}(-1)^{m}[F(x)]^{m},
$$

the pdf of the t^{th} order statistic can be expressed as

$$
f_{t:n}(x) = \frac{n!f(x)}{(t-1)!(n-t)!}[F(x)]^{t-1}[1-F(x)]^{n-t}
$$

$$= \frac{n!f(x)}{(t-1)!(n-t)!}\sum_{m=0}^{n-t}(-1)^m\binom{n-t}{m}[F(x)]^{m+t-1}. \quad (9)$$

Based on (1) and (2), we can write

$$\begin{aligned} f(x)F(x)^{m+t-1} &= \alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1} g(x;\xi) \\ &\times \exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)(1-G(x;\xi))^{-2} \\ &\times\left(1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right)^{m+t-1}. \end{aligned}$$

Following the same steps of the density expansion leading to (5), we get

$$f(x)F(x)^{m+t-1} = \sum_{w=0}^{\infty} a_{w+1}h_{w+1}(x;\xi), \quad (10)$$

where

$$\begin{aligned} a_{w+1} &= \alpha\beta\sum_{i,q,j,p,l,k,v=0}^{\infty}\binom{m+t-1}{v}\binom{\beta-1}{q}\binom{i}{j}\binom{\beta j}{p}\frac{\Gamma(\alpha(p+q+1)+1+l)}{\Gamma(\alpha(p+q+1)+1)l!} \\ &\times\binom{k}{w}\frac{\Gamma(\alpha(q+p+1)+l)}{\Gamma(\alpha(q+p+1)+l-k)k!}\frac{(-1)^{j+k+w}}{i!}\frac{(-1)^v(v+1)^i}{w+1}. \end{aligned}$$

Substituting (10) into (9), we obtain

$$f_{t:n}(x) = \frac{n!}{(t-1)!(n-t)!}\sum_{w=0}^{\infty}\sum_{m=0}^{n-t}(-1)^m\binom{n-t}{m}a_{w+1}h_{w+1}(x;\xi),$$

where $h_{w+1}(x;\xi) = (w+1)[G(x;\xi)]^w g(x;\xi)$ is the exponentiated-G (E-G) pdf with the power parameter $w+1>0$ and parameter vector ξ.

3.5 Rényi entropy

Rényi entropy (Rényi, 1960) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^{\infty}[f(x;\alpha,\beta,\xi)]^v dx\right), v \neq 1, v>0.$$

Rényi entropy tends to Shannon entropy as $v \to 1$.

Rényi entropy for the OPGW-G family of distributions is given by

$$I_R(v) = \frac{1}{1-v}\log\left[\sum_{i,j,p,q,k,w=0}^{\infty} w_{i,j,p,q,k,w}\exp((1-v)I_{REG})\right],$$

for $v>0$, $v\neq 1$, where $I_{REG}=\int_0^\infty\left(\left[\frac{w}{v}+1\right](G(x;\xi))^{\frac{w}{v}}g(x;\xi)\right)^v dx$ is the Rényi entropy of E-G distribution with power parameter $\frac{w}{v}+1$, and

$$\begin{aligned} w_{i,j,p,q,k,w} &= (\alpha\beta)^v\frac{v^i}{i!}\binom{v(\beta-1)}{j}\binom{i}{p}\binom{\beta p}{q}\frac{\Gamma(v+\alpha(q+j+v)+l)}{\Gamma(v+\alpha(q+j+v))l!}\\ &\quad\times(-1)^{p+k+w}\binom{\alpha(j+q+v)+l-v}{k}\binom{k}{w}\frac{1}{\left[\frac{w}{v}+1\right]^v}. \end{aligned}$$

See the appendix for the expansions leading to the Rényi entropy for the OPGW-G family of distributions given above.

4 Estimation

We assume that X follows OPGW-G family of distributions and let $\mathbf{\Delta}=(\alpha,\beta,\xi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n=\ell_n(\mathbf{\Delta})$ based on a random sample of size n from the OPGW-G family of distributions is given by

$$\begin{aligned} \ell_n(\mathbf{\Delta}) &= n\ln(\alpha\beta)+(\alpha-1)\sum_{i=1}^n\ln\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)\\ &\quad+\sum_{i=1}^n\left(1-\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^\alpha\right]^\beta\right)-2\sum_{i=1}^n\ln\left(1-G(x_i;\xi)\right)\\ &\quad+(\beta-1)\sum_{i=1}^n\ln\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^\alpha\right]+\sum_{i=1}^n\ln(g(x_i;\xi)). \end{aligned}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\mathbf{\Delta}=(\alpha,\beta,\xi)^T$, that is, elements of the score vector $U(\mathbf{\Delta})$ are given in the appendix.

Setting the nonlinear system of equations $(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\xi_k})^T=\mathbf{0}$, and solving them simultaneously yields the maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$. It is usually more convinient to adopt nonlinear optimization method such as Newton-Raphson procedure to maximize ℓ numerically. We maximize the likelihood function using NLmixed in SAS as well as the function nlm in R (rdevelopment-coreteam, 2011).

Let $J(\hat{\mathbf{\Delta}})$ be the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$. Under standard regulatory conditions when $n\to\infty$, the distribution of $\hat{\mathbf{\Delta}}$ can be approximated by a multivariate normal $N_{q+2}(\underline{0},J(\hat{\mathbf{\Delta}})^{-1})$ distribution to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions including Seregin (2010), Santos Silva and Tenreyro (2010), Zhou (2009), and Xia et al. (2009). At this point we are not able to address the theoretical aspects (existence, uniqueness) of the MLE of the parameters of the OPGW-G family of distributions. This is because the maximum likelihood estimates are obtained by numerical approximations, as no explicit expression for the solution of the likelihood equations exists. Also, it is noted that this could be a subject of further research as it was not considered in this paper.

The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2\log(L)$), Akaike information criterion ($AIC = 2p - 2\log(L)$), Bayesian information criterion ($BIC = p\log(n) - 2\log(L)$) and consistent Akaike information criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented. In order to compare the models, we use the criteria stated above. We also obtain the goodness-of-fit statistics: Crameŕ-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by Chen and Balakrishnan (1995), as well as Kolmogorov-Smirnov (KS) statistic and its p-value. Note that for for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and KS, smaller values are preferred.

5 Simulation study

The performance of the OPGW-LLoG distribution is examined by conducting various simulations for different sizes (n= 50, 100, 200, 400, 800) via the R package. We simulate $N = 1000$ samples for the true parameters values given in Table 1. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively. From the results, we can clearly verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

6 Applications

Below we present examples to illustrate the flexibility and usefulness of the OPGW-LLoG distribution for data modeling. The OPGW-LLoG distribution is fitted to the data sets and these fits are compared to the fits of the non-nested, the new power generalized Weibull-log-logistic (NPGW-LLoG) distribution with ($s = 1$) by Oluyede et al. (2020), Marshall-Olkin Log-logistc (MOLLD) distribution (Wenhao, 2013), exponentiated log-logistic (ELLoG) distribution by Rosaiah et al. (2007), extended log-logistic (ExLLoG) distribution by Lima and Cordeiro (2017), the Topp-Leone generalized exponential (TLGE) distribution by Sangsanit and Bodhisuwan (2016), the inverse-power logistic-exponential (IPLE) distribution by Sobhi and Mashail (2020) and Weibull exponential (WE) distribution by Oguntunde et al. (2015). The Pdf's of models of comparison are given in the appendix.

Table 1: Monte Carlo simulation results for OPGW-LLoG distribution: Mean, RMSE and average bias

		$\alpha=1.0,\beta=1.0,c=0.1$			$\alpha=1.5,\beta=1.0,c=1.0$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	50	1.053	0.486	0.053	1.756	2.149	0.256
	100	1.049	0.327	0.049	1.598	0.277	0.098
α	200	1.036	0.236	0.036	1.533	0.138	0.033
	400	1.008	0.088	0.008	1.511	0.095	0.011
	800	1.003	0.055	0.003	1.497	0.067	-0.002
	50	1.839	2.265	0.839	6.590	10.594	5.590
	100	1.589	1.909	0.589	4.770	7.702	3.770
β	200	1.441	1.628	0.441	3.980	6.179	2.980
	400	1.287	1.253	0.287	2.403	3.689	1.403
	800	1.163	0.919	0.163	1.585	2.056	0.585
	50	0.360	1.540	0.260	1.016	0.931	0.016
	100	0.220	0.398	0.120	0.950	0.674	-0.049
c	200	0.219	0.356	0.119	0.873	0.546	-0.126
	400	0.173	0.252	0.073	0.924	0.422	-0.075
	800	0.143	0.177	0.043	0.944	0.306	-0.055
		$\alpha=1.5,\beta=1.0,c=1.5$			$\alpha=1.0,\beta=0.1,c=1.0$		
	50	1.760	2.068	0.260	1.448	3.116	0.448
	100	1.598	0.262	0.098	1.137	0.696	0.137
α	200	1.545	0.149	0.045	1.130	0.517	0.130
	400	1.521	0.098	0.021	1.073	0.331	0.073
	800	1.503	0.066	0.003	1.028	0.156	0.028
	50	6.713	10.691	5.713	0.308	0.893	0.208
	100	4.728	7.635	3.728	0.277	0.657	0.177
β	200	3.780	6.005	2.780	0.198	0.401	0.098
	400	2.237	3.445	1.237	0.159	0.228	0.059
	800	1.565	2.034	0.565	0.151	0.194	0.051
	50	1.501	1.428	0.001	1.510	2.282	0.510
	100	1.425	0.975	-0.074	1.363	1.793	0.363
c	200	1.345	0.807	-0.154	1.352	1.668	0.352
	400	1.413	0.616	-0.086	1.278	1.313	0.278
	800	1.428	0.452	-0.071	1.245	1.181	0.245

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al., 1983) are given in Figure 5 and Figure 7. For the probability plot, we plotted $F(x_{(j)};\hat{\alpha},\hat{\beta},\hat{\xi})$ against $\dfrac{j-0.375}{n+0.25}$, $j=1,\,2,\,\cdots,\,n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS=\sum_{j=1}^{n}\left[F(x_{(j)};\hat{\alpha},\hat{\beta},\hat{\xi})-\left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan (1995) are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

The estimates of the parameters of OPGW-LLoG distribution (standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (KS) and its p-value as well as SS are given in Tables 2-5.

6.1 Survival Data

The data corresponds to the survival times (in years) of a group of patients given chemotherapy treatment reported by Bekker et al. (2000), The data are:
0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

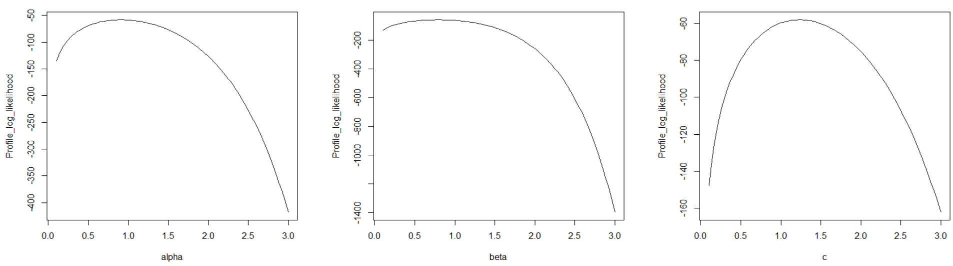

Figure 4: Profile likelihood function plots for parameters of OPGW-LLoG on the survival data set

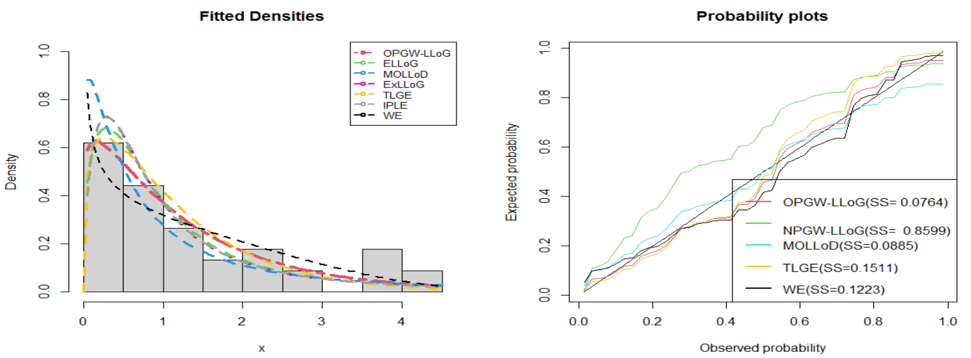

Figure 5: Fitted densities and probability plots of the survival data

From Table 3, the AIC, AICC, BIC, SS, W^*, A^*, under OPGW-LLoG distribution are the smallest than any other distribution. Additionaly, the p-value of KS-test under OPGW-LLoG distribution is greater than all of the non-nested models. For these reasons, the OPGW-LLoG distribution is more appropriate for fitting this data.

6.2 Time to failure of Kevlar 49/epoxy Data

The data set consists of 101 observations of stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed, so that the complete data set with the exact times of failure is recorded. These failure times in hours, are originally given by Barlow et al. (1984). The data are

Table 2: Estimates of models for survival data

Model	Estimates		
	α	β	C
OPGW-LLoG	0.911	0.796	1.234
	(0.080)	(0.095)	(0.059)
	α	β	c
NPGW-LLoG	2.684	0.874	0.554
	(0.450)	(0.122)	(0.100)
	α	β	a
ELLoG	1.040	1.646	0.803
	(1.328)	(0.908)	(1.037)
	α	β	γ
MOLLD	1.703×10^{03}	1.124	1.865×10^{-04}
	(2.6×10^{-07})	(1.3×10^{-01})	(2.0×10^{-04})
	α	β	λ
ExLLoG	1.507	1.316	2.538×10^{-05}
	(1.8×10^{-01})	(3.4×10^{-01})	(2.1×10^{-02})
	α	λ	β
TLGE	6.8×10^{-03}	1.000	1.9×10^{02}
	(1.3×10^{-03})	(1.6×10^{-01})	(4.6×10^{-05})
	α	β	λ
IPLE	2.7×10^{02}	4.1×10^{-03}	6.9×10^{-01}
	(7.6×10^{-09})	(4.9×10^{-04})	(5.2×10^{-04})
	α	β	λ
WE	0.878	0.747	0.500
	(0.762)	(0.202)	(0.367)

Table 3: Goodness-of-fit statistics for survival data

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
OPGW-LLoG	116.220	122.220	122.806	127.640	0.075	0.509	0.105	.663	0.076
NPGW-LLoG	128.843	134.842	135.428	140.262	0.086	0.623	0.231	.013	0.859
ELLoG	120.067	126.067	126.652	131.487	1.826	9.415	0.544	.000	2.071
MOLLD	125.545	131.545	132.131	136.965	0.076	0.515	0.143	.284	0.088
ExLLoG	120.371	126.371	126.956	131.791	15.945	90.332	0.987	.000	15.51
TLGE	117.532	123.532	124.117	128.952	0.079	0.532	0.125	.445	0.151
IPLE	120.385	126.384	126.970	131.804	15.943	90.331	0.987	.000	15.49
WE	117.369	123.369	123.954	128.789	0.135	0.882	0.122	.472	0.122

0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

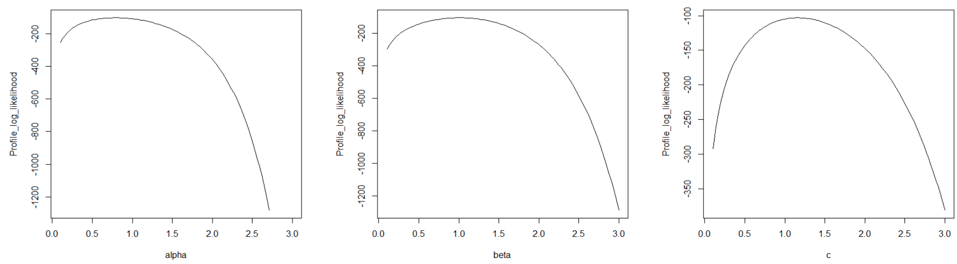

Figure 6: Profile likelihood function plots for parameters of OPGW-LLoG on the Kevlar 49/epoxy data

The values in Table 5, show that the OPGW-LLoG distribution has the smallest values for AIC, AICC, BIC, SS, W^*, A^*, KS and the largest p-value compared to all fitted distributions. From that, we conclude that the OPGW-LLoG distribution provides the "best" fit as compared to other fitted models for Kevlar 49/epoxy data set.

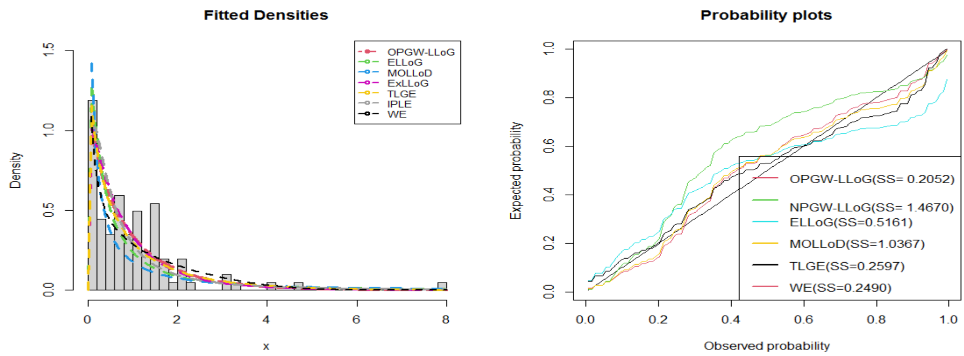

Figure 7: Fitted densities and probability plots of Kevlar 49/epoxy data

7 Conclusions

We propose a new generalized distribution called the odd power generalized Weibull-G (OPGW-G) family of distributions. The statistical properties of the new family such as moments, incomplete moments, probability weighted moments, distribution of order statistics and entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the special case of the OPGW-G was examined by conducting various simulations for different sample sizes. Finally, the special case of the OPGW-G named OPGW-LLoG is fitted to real data sets to illustrate the potentiality of the proposed family of distributions.

Table 4: Estimates of models for time to failure of Kevlar 49/epoxy data

Model	Estimates		
	α	β	C
OPGW-LLoG	1.031	1.013	0.892
	(0.038)	(0.070)	(0.044)
	α	β	c
NPGW-LLoG	2.007	0.677	0.735
	(0.281)	(0.070)	(0.137)
	α	β	a
ELLoG	0.512	1.000	1.120
	(0.285)	(0.162)	(0.407)
	α	β	γ
MOLLD	5.0×10^{05}	7.4×10^{-01}	3.9×10^{-05}
	(3.5×10^{-16})	(3.2×10^{-09})	(6.1×10^{-06})
	α	β	λ
ExLLoG	1.040	0.125	9.613
	(0.121)	(0.125)	(8.159)
	α	λ	β
TLGE	6.203	0.250	0.206
	(10.182)	(0.078)	(0.227)
	α	β	λ
IPLE	7.9×10^{02}	1.1×10^{-03}	6.9×10^{-01}
	(1.5×10^{-10})	(9.6×10^{-05})	(1.6×10^{-04})
	α	β	λ
WE	1.176	0.619	0.500
	(0.513)	(0.120)	(0.212)

Table 5: Goodness-of-fit statistics for time to failure of Kevlar 49/epoxy data

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
OPGW-LLoG	205.926	211.926	212.173	219.771	0.196	1.102	0.094	.333	0.205
NPGW-LLoG	231.327	237.327	237.574	245.172	0.631	3.403	0.234	.000	1.467
ELLoG	232.419	238.419	238.666	246.264	0.622	3.367	0.142	.033	0.516
MOLLD	259.068	265.068	265.315	272.913	0.609	3.279	0.201	.001	1.036
ExLLoG	208.816	214.816	215.063	222.661	34.01	201.19	0.993	.000	34.37
TLGE	209.485	215.485	215.733	223.331	0.296	1.591	0.103	.228	0.259
IPLE	225.409	231.409	231.656	239.254	33.68	199.57	0.994	.000	32.94
WE	220.8034	226.8034	227.050	234.648	0.182	1.259	0.116	.126	0.249

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Appendix

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