Journal of Statistical Modelling: Theory and Applications
Vol. 2, No. 1, 2021, pp. 151-166
Yazd University Press 2021

JSMTA

Research Paper

Improvement of the mixed Liu estimator applying Jackknife method in linear regression models

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Received: September 14, 2020/ Revised: August 30, 2021 / Accepted: September 12, 2021

Abstract: In the presence of multicollinearity in the regression models, the ordinary least squares estimator loses its performance. Some solutions to reduce the effects of multicollinearity have been proposed, including the application of biased estimators such as Liu estimate and estimation under linear restrictions. But due to the Liu estimator being biased, the Jackknife method has been introduced to reduce the bias. In this paper, we will examine the Jackknifed Liu estimator and propose a new estimator under stochastic linear restrictions namely stochastic restricted Jackknifed Liu estimator. A simulation study is conducted to investigate the performance of this new estimator using two measures namely mean squared errors and absolute bias. From simulation study results, we find that the new estimator outperforms the OLS and Liu estimators, and it is superior to both OLS and Liu estimators, using the mean squared errors and absolute bias criteria.

Keywords: Jackknifed Liu estimator; Multicolinearity; Pseudo-values; Stochastic linear restrictions.
Mathematics Subject Classification (2010): 62J05

1 Introduction

Multicollinearity occurs when two or more explanatory variables are highly correlated with each other. For instance, the effects of multicollinearity can be reduced by increasing the sample size, removing the correlated variables from the model, and adding

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auxiliary information. The concept of multicollinearity, the identification methods, strategies to deal with this problem, and its implications have been widely discussed by Montgomery et al (2001), Daoud (2017), and Whinship and Western (2016). On the other hand, some solutions proposed for multicollinearity leads to biased estimators. These estimators reduce the variance of the resulting estimates significantly by accepting some bias. These estimators include shrinkage estimator, Stein estimator (Stein, 1956), Ridge estimator (Horel and Kennrad, 1970), and Liu estimator (Liu, 1993).

One of the biased estimators that are widely used in practical work is the Liu estimator. This estimator was first introduced by Liu (1993). In this method, Liu obtained a new form of the estimator by combining Stein and Ridge estimators. Numerous studies have been conducted on this estimator. Sakallioglu et al (2001) compared the ridge and the Liu estimators under the mean square errors (MSEs) criterion. Akdeniz and Erol (2003) also made a comparison between the generalized Ridge estimator and the generalized Liu estimator.

Other forms of Liu estimator have also been introduced by other researchers. Yang et al (2009), by combining the weighted mixed and Liu estimators introduced the weighted Liu estimator. Akdenizet al (2009) proposed a Liu-type estimator for the vector of regression coefficients in a semi-parametric linear regression model. Alheety and Kibria (2009) proposed an almost unbiased Liu estimator in regression models with heteroscedastic errors. In recent years, Gruber (2010) has presented the two-parameter Liu estimator as a Bayesian and Minimax estimator. Liu (2011) proposed an improved Liu estimator based on the sum of squares prediction errors criterion, and Li and Yang (2012) proposed a new form of Liu-type. Wu and Asar (2017) also studied Liu estimator under exact linear constraints in logistic regression models.

On the other hand, sometimes a set of prior information about regression coefficients is available and their use in regression analysis and estimation process can significantly reduce the effects of multicollinearity. This prior information is usually expressed in the form of linear constraints. Applying these constraints on the regression coefficients vector reduces the variance and improves the accuracy of estimates and thus decreases the multicollinearity effects. Ozbay and Kacirenlar (2018) proposed a new two parameter-weighted mixed estimator in a linear regression model with stochastic linear restrictions and conducted a comparison study to investigate the performance of the new estimator in comparison with other estimators. The idea of using prior information was first introduced by Durbin (1953). He simultaneously used sample information and prior information in the regression model. Subsequently, Thiel and Goldberger (1961) developed a new estimator called the mixed estimator by integrating the prior information and the sample data.

Also, many researchers have emphasized that the simultaneous use of these two approaches, namely the applications of the biased estimator and prior information, can further reduce the effects of multicollinearity (Yang and Xu, 2009 and Hubert and Wijekoon, 2006). Therefore, using these two solutions simultaneously in a regression model seems to be useful to reduce the effects of multicollinearity.

Kacirnelar et al. (1999) used the idea of Sarkar (1992) by combining the Liu and the restricted least squares estimators and proposed a restricted Liu estimator. Kacirenlar (2001) compared the restricted Liu and the restricted least squares estimators. The idea of a mixed estimator was motivated by Hubert and Wijekoon (2006) to introduce a

Liu estimator under the stochastic linear restrictions to obtain a new estimator namely a stochastic restricted Liu estimator.

The application of biased estimators is considered to be the most common method for reducing the effects of multicollinearity. In this regard, many efforts have been made to provide a suitably biased estimator. Although these estimators are biased, the results from their fitting are more inferred. Therefore, the only issue of concern regarding the application of these estimators is their biasedness. So, the researchers introduce some methods to reduce the bias of these estimators. One of these methods is the Jackknife technique. In addition to being a nonparametric bias reduction method, the Jackknife technique has also been applied as a method for estimating variance and identifying influential observations in a data set. The Jackknife technique was first introduced by Quenouille (1949, 1956) to reduce the bias in the time series models. Then Tukey (1958) was able to estimate the variance using the Jackknife technique. Cook (1977) used this method to represent influential observations on a data set in the context of diagnostics.

Hinkley (1977) introduced a weighted version of the Jackknife technique in unbalanced models. After that to reduce the bias of the Ridge estimator, Sing et al. (1986) used the Jackknife technique and proposed Jackknifed Ridge estimator, and then evaluated its bias under the MSE criterion in comparison with some estimators. Nyquist (1988) also presented the application of the Jackknife technique in Ridge regression.

Recently, Akdeniz and Akdeniz (2012) following Sing et al. (1986) applied the Jackknife technique for Liu estimator and obtained an estimator which is very similar to the almost unbiased Liu estimator introduced by Akdeniz and Kacirenlar (1995). The performance of the new estimator was investigated in comparison with other estimators.

In this study, we introduce a form of stochastic restricted Liu estimator using Liu estimator and linear restrictions in the form of stochastic and we study the performance of this estimator is compared with some other estimators. Also to reduce the bias of the new estimator, we apply the Jackknife technique and obtain another estimator. We will call this new estimator the mixed Jackknifed Liu estimator and show that it has better performance than Liu estimator under MSE and absolute bias criteria.

The article is organized as follows: in section 2 the model and estimators are presented. Then the Jackknifed technique and Jackknifed Liu estimator will be presented in section 3. Section 4 will focus on the application of Linear restrictions and two new estimators are proposed. A simulation study is performed to investigate the performance of the new estimator under the MSEs and the absolute bias criteria in section 5. Section 6 considers a real data set to show the applicability of the mixed Jackknifed Liu estimator. The paper ends with some conclusions in section 7.

2 Model and estimators

Consider the following regression model

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} \sim N(0, \sigma^2 \boldsymbol{I}_n), \tag{1}$$

where X is an $n \times p$ matrix of explanatory variables, Y is an $n \times 1$ vector of observations on the response variable, β is a $p \times 1$ vector of unknown parameters, ε is an $n \times 1$ vector of independent and identically distributed random errors with mean 0 and variance-covariance matrix $\sigma^2 I$.

It is known that Model (1) can be written in a canonical form. Let $\boldsymbol{T}$ is an orthogonal matrix such that

$$\boldsymbol{T}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{T} = \boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is a $p \times p$ diagonal matrix whose elements are eigenvalues of $\boldsymbol{X}'\boldsymbol{X}$. Using $\boldsymbol{T}$ matrix; we obtain the canonical form of model (1) as

$$\boldsymbol{Y} = \boldsymbol{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \tag{2}$$

where $\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{T}$ and $\boldsymbol{\gamma} = \boldsymbol{T}'\boldsymbol{\beta}$. Therefore, the least squares estimator for model (2) is obtained as

$$\hat{\gamma} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{Y} = \boldsymbol{\Lambda}^{-1}\boldsymbol{Z}'\boldsymbol{Y}. \tag{3}$$

The ordinary least squares estimator is unbiased and has proven its performance in many practical applications. But when there is multicollinearity among two or more explanatory variables, the ordinary least squares are unstable, and such estimation cannot be relied upon in practical work because the results would be unreliable.

To overcome the problem of multicollinearity, Liu (1993) introduced a biased estimator namely, the Liu estimator. To achieve this estimator, he combined some information in the form of stochastic linear restrictions, called Liu's constraints, with the model (1). Therefore, the Liu estimator of vector $\boldsymbol{\gamma}$ in the model (2) is given by

$$\hat{\gamma}(d) = (\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1}(\boldsymbol{Z}'\boldsymbol{Y} + d\hat{\gamma}) = (\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1}(\boldsymbol{\Lambda} + d\boldsymbol{I}_p)\hat{\gamma} = \boldsymbol{F}_d\hat{\gamma}, \tag{4}$$

where $0 < d < 1$ is called Liu parameter and

$$\boldsymbol{F}_d = (\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1}(\boldsymbol{\Lambda} + d\boldsymbol{I}_p) = \operatorname{diag}\left\{\frac{\lambda_1 + d}{\lambda_1 + 1}, \ldots, \frac{\lambda_p + d}{\lambda_p + 1}\right\}.$$

Since $\boldsymbol{\gamma} = \boldsymbol{T}'\boldsymbol{\beta}$, so the least squares and the Liu estimators of vector $\boldsymbol{\beta}$ in model (1) will be $\hat{\boldsymbol{\beta}} = \boldsymbol{T}\hat{\gamma}$ and $\hat{\boldsymbol{\beta}}(d) = \boldsymbol{T}\hat{\gamma}(d)$, respectively.

The bias vector and the variance-covariance matrix of the Liu estimator are given

$$\begin{aligned} Bias(\hat{\gamma}(d)) &= E(\hat{\gamma}(d)) - \boldsymbol{\gamma} = \boldsymbol{F}_d\gamma - \gamma = (\boldsymbol{F}_d - \boldsymbol{I}_p)\gamma \\ &= -(\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1}(\boldsymbol{I}_p - d\boldsymbol{I}_p)\gamma, \\ Var(\hat{\gamma}(d)) &= \sigma^2(\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1}(\boldsymbol{\Lambda} + d\boldsymbol{I}_p)\boldsymbol{\Lambda}^{-1}(\boldsymbol{\Lambda} + d\boldsymbol{I}_p)(\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1} \\ &= \sigma^2\boldsymbol{F}_d\boldsymbol{\Lambda}^{-1}\boldsymbol{F}_d. \end{aligned}$$

To apply this estimator in practice, we face the problem of choosing an unknown parameter d. Liu (1993) outlines several methods for estimating d, in which only the minimization of MSE of Liu estimator is considered here. By minimizing the MSE of Liu estimator we obtain an estimator of d as

$$d = \sum_{j=1}^{p} \frac{\gamma_j^2 - \sigma^2}{(\lambda_j + 1)^2} \Bigg/ \sum_{j=1}^{p} \frac{\sigma^2 + \lambda_j\gamma_j^2}{(\lambda_j + 1)^2\lambda_j}.$$

Substituting γ_j's and σ^2 by their unbiased estimators, we get the estimator of d as

$$\hat{d}_{mm} = 1 - \hat{\sigma}^2 \left(\sum_{j=1}^{p} \frac{1}{\lambda_j(\lambda_j+1)} \Big/ \sum_{j=1}^{p} \frac{\hat{\gamma}_j^2}{(\lambda_j+1)^2} \right). \quad (5)$$

See Liu (1993) for other methods of selection d.

3 Jackknife technique and Jackknifed Liu estimator

The Jackknife technique or ”case deletion” is a cross-validation method first introduced by Quenouille (1956) as a nonparametric bias reduction method. Then the Jackknife method was used for nonparametric variance estimation by Tukey (1958).
More recently many researchers have applied the Jackknife technique on some biased estimators to reduce related bias. Akdeniz et al. (2012) proposed a new estimator called Jackknifed Liu by applying the Jackknife method on the Liu estimator which reduces the bias of the Liu estimator. In doing so, they first obtained Liu estimator for the canonical model (2) without ith observation, which yields the following result

$$\hat{\gamma}_{(-i)}(d) = \hat{\gamma}(d) - \frac{\boldsymbol{A}^{-1}\boldsymbol{z}_i e_i^*(d)}{1-w_i^*},$$

where $\boldsymbol{z}_i$ is the ith row of $\boldsymbol{Z}$ matrix, and $\hat{\gamma}_{(-i)}(d)$ is the Liu estimator for the parameters vector of the canonical model (2) when ith observation is not considered. Also $w_i^* = \boldsymbol{z}_i' A^{-1}\boldsymbol{z}_i$ is the ith diagonal element of $\boldsymbol{W}^* = \boldsymbol{Z}\boldsymbol{A}^{-1}\boldsymbol{Z}'$ and $e_i^*(d) = y_i - \boldsymbol{z}_i'\hat{\gamma}(d)$ is the ith residual obtaining from fitting the Liu estimator for model (2). Matrix $\boldsymbol{A}$ is defined as

$$\boldsymbol{A}^{-1} = (\boldsymbol{\Lambda} + \boldsymbol{I})^{-1}(\boldsymbol{I} + \boldsymbol{d}\boldsymbol{\Lambda}^{-1}) = \boldsymbol{F}_d\boldsymbol{\Lambda}^{-1},$$

where $F_d = (\boldsymbol{\Lambda} + \boldsymbol{I}_p)^{-1}(\boldsymbol{\Lambda} + \boldsymbol{d})$. Hence, the weighted pseudo-values are defined as follows

$$\tilde{Q}_i = \hat{\gamma}(d) + n(1-w_i^*)(\hat{\gamma}(d) - \hat{\gamma}_{(-i)}(d)),$$

(See Hinkley, 1977 and Nayquist, 1988). Therefore, the Jackknifed Liu estimator for model (2) is given by

$$\tilde{\gamma}(d) = (2\boldsymbol{I}_p - \boldsymbol{F}_d)\boldsymbol{F}_d\hat{\boldsymbol{\gamma}}.$$

Since $\boldsymbol{\gamma} = \boldsymbol{T}'\boldsymbol{\beta}$, therefore $\tilde{\boldsymbol{\beta}}(d) = \boldsymbol{T}\tilde{\boldsymbol{\gamma}}(d)$ is the Jackknifed Liu estimator for $\boldsymbol{\beta}$ of the model (1).

4 Jackknifed Liu estimator under stochastic linear restrictions

If prior information on the vector of regression coefficients is available, they are usually expressed in the form of linear constraints, which are imposed on coefficient vector in two forms as stochastic and exact linear restrictions. The previous studies show that using this information about the vector of regression parameters reduces the MSE of the estimator. So in this section, we will focus on how to use prior information and show how combining this information will improve the estimators. Also, according to the used method by Ozkale (2009) and Li and Yang (2010), by integrating the prior information in the form of linear given observation restrictions in Liu Jackknifed estimation procedure, we introduce a new estimator called the mixed Jackknifed Liu estimator.

Let us be given some prior information about $\boldsymbol{\beta}$ in the form of a set of m independent stochastic linear restrictions as

$$\boldsymbol{r} = \boldsymbol{R}\boldsymbol{\beta} + \boldsymbol{\phi}, \qquad \boldsymbol{\phi} \sim (0, \sigma^2 \boldsymbol{I}_m), \tag{6}$$

where $\boldsymbol{r}$ is an $m \times 1$ vector of known values, $\boldsymbol{R}$ is an $m \times p$ known matrix with rank m, and $\boldsymbol{\phi}$ an $m \times 1$ vector of disturbances with mean 0 and variance-covariance matrix $\sigma^2 \boldsymbol{I}_m$. Also, we assume that the elements of $\boldsymbol{\phi}$ are independent of the elements of $\boldsymbol{\varepsilon}$ vector.

To estimate regression coefficients under stochastic linear restrictions, we combine model (1) and prior information (6) as follows

$$\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{R} \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\phi} \end{pmatrix}. \tag{7}$$

Using new notation, $\tilde{\boldsymbol{Y}} = \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{r} \end{pmatrix}$, $\tilde{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{R} \end{pmatrix}$, $\tilde{\varepsilon} = \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\phi} \end{pmatrix}$, we obtain the following model

$$\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{X}}\boldsymbol{\beta} + \tilde{\varepsilon}, \qquad \tilde{\varepsilon} \sim (0, \sigma^2 \boldsymbol{I}). \tag{8}$$

It is known that the model (8) can be written in canonical form. Hence there exist a matrix $\boldsymbol{P}_{p\times p}$ such that

$$\boldsymbol{P}'\tilde{\boldsymbol{X}}'\tilde{\boldsymbol{X}}\boldsymbol{P} = \boldsymbol{\Pi}, \qquad \boldsymbol{P}'\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}_p,$$

where $\boldsymbol{\Pi}$ is a diagonal matrix whose elements are eigenvalues of the matrix $\tilde{\boldsymbol{X}}'\tilde{\boldsymbol{X}} = \boldsymbol{X}'\boldsymbol{X} + \boldsymbol{R}'\boldsymbol{R}$. Using matrix $\boldsymbol{P}$, the model (8) is rewritten as

$$\tilde{\boldsymbol{Y}} = \boldsymbol{U}\boldsymbol{\theta} + \tilde{\varepsilon}, \tag{9}$$

where $\boldsymbol{U} = \tilde{\boldsymbol{X}}\boldsymbol{P}$ and $\boldsymbol{\theta} = \boldsymbol{P}'\boldsymbol{\beta}$. Using the least squares method for model (9), the *OLS* estimator under restriction (6) is obtained as follows

$$\hat{\boldsymbol{\theta}}_m = (\boldsymbol{U}'\boldsymbol{U})^{-1}\boldsymbol{U}'\tilde{\boldsymbol{Y}} = \boldsymbol{\Pi}^{-1}\boldsymbol{U}'\tilde{\boldsymbol{Y}}.$$

This estimator is called the mixed estimator. We can show that the mixed estimator has a smaller variance than the ordinary least squares estimator. See Rao et al (2008) for more details.

Now by integrating the Liu constraints with the canonical model (9), we can obtain the Liu estimator under stochastic linear restrictions. So, we have

$$\hat{\boldsymbol{\theta}}_m(d) = (\boldsymbol{U}'\boldsymbol{U} + \boldsymbol{I})^{-1}(\boldsymbol{U}'\boldsymbol{U} + d\boldsymbol{I})\hat{\boldsymbol{\theta}}_m = (\boldsymbol{\Pi} + \boldsymbol{I})^{-1}(\boldsymbol{\Pi} + d\boldsymbol{I})\hat{\boldsymbol{\theta}}_m = \boldsymbol{G}_d\hat{\boldsymbol{\theta}}_m,$$

where $\boldsymbol{G}_d$ is a diagonal matrix whose elements are defined as follows

$$\boldsymbol{G}_d = \text{diag}\left\{\frac{\pi_1 + d}{\pi_1 + 1}, \ldots, \frac{\pi_p + d}{\pi_p + 1}\right\}.$$

Also, the bias vector and variance-covariance matrix of mixed Liu estimator is given as

$$\begin{aligned}
Bias(\hat{\boldsymbol{\theta}}_m(d)) &= E(\hat{\boldsymbol{\theta}}_m(d)) - \boldsymbol{\theta} = \boldsymbol{G}_d\boldsymbol{\theta} - \boldsymbol{\theta} = (\boldsymbol{G}_d - \boldsymbol{I})\boldsymbol{\theta}, \\
Var(\hat{\boldsymbol{\theta}}_m(d)) &= \sigma^2 \boldsymbol{G}_d \Pi^{-1} \boldsymbol{G}_d, \\
MSE(\hat{\boldsymbol{\theta}}_m(d)) &= Var(\hat{\boldsymbol{\theta}}_m(d)) + Bias(\hat{\boldsymbol{\theta}}_m(d))[Bias(\hat{\boldsymbol{\theta}}_m(d))]' \\
&= \sigma^2 \boldsymbol{G}_d \Pi^{-1} \boldsymbol{G}_d + (\boldsymbol{G}_d - \boldsymbol{I}_p)\boldsymbol{\theta}\boldsymbol{\theta}'(\boldsymbol{G}_d - \boldsymbol{I}_p).
\end{aligned}$$

The mixed Liu estimator is biased, so by applying the Jackknife method, the bias of this estimator will be reduced. We use the method proposed by Akdeniz and Akdeniz (2012) to obtain Jackknifed Liu estimator under stochastic linear restrictions. Consider the $\hat{\boldsymbol{\theta}}_m(d)$ estimator for the canonical model (9). The Jackknife technique is based on observation deletion, so we obtain the $\hat{\boldsymbol{\theta}}_m(d)$ estimator after eliminating the ith observation, $\hat{\boldsymbol{\theta}}_{m(-i)}(d)$. Defining $\boldsymbol{XP} = \tilde{\boldsymbol{Z}}$ and $\boldsymbol{RP} = \tilde{\boldsymbol{R}}$, we have

$$\begin{aligned}
\hat{\boldsymbol{\theta}}_{m(-i)}(d) =& (\boldsymbol{B} - \tilde{\boldsymbol{z}}_i\tilde{\boldsymbol{z}}_i')^{-1}(\tilde{\boldsymbol{Z}}'\boldsymbol{Y} - \tilde{\boldsymbol{z}}_i y_i + \tilde{\boldsymbol{R}}'\boldsymbol{r}) \\
=& \left(\boldsymbol{B}^{-1} + \frac{\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i\tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}}{1 - \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i}\right)(\tilde{\boldsymbol{Z}}'\boldsymbol{Y} + \tilde{\boldsymbol{R}}'\boldsymbol{r} - \tilde{\boldsymbol{z}}_i y_i) \\
=& \boldsymbol{B}^{-1}(\tilde{\boldsymbol{Z}}'\boldsymbol{Y} + \tilde{\boldsymbol{R}}'\boldsymbol{r}) - \boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i y_i \\
&+ \frac{\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i\tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}}{1 - \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i}(\tilde{\boldsymbol{Z}}'\boldsymbol{Y} + \tilde{\boldsymbol{R}}'\boldsymbol{r}) - \frac{\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i\tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i y_i}{1 - \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i} \\
=& \hat{\boldsymbol{\theta}}_m(d) - \boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i y_i\left(1 + \frac{\tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i}{1 - \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i}\right) + \frac{\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i\tilde{\boldsymbol{z}}_i'\hat{\boldsymbol{\theta}}_m(d)}{1 - \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i} \\
=& \hat{\boldsymbol{\theta}}_m(d) - \frac{\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i\left(y_i - \tilde{\boldsymbol{z}}_i'\hat{\boldsymbol{\theta}}_m(d)\right)}{1 - \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i} \\
=& \hat{\boldsymbol{\theta}}_m(d) - \frac{\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i\tilde{e}_i(d)}{1 - \tilde{w}_i}.
\end{aligned}$$

The value $\hat{\boldsymbol{\theta}}_{m(-i)}(d)$ is the mixed Liu estimator when the ith observation is removed from the data set. $\tilde{\boldsymbol{Z}}_{(-i)}$ and $\boldsymbol{Y}_{(-i)}$ are matrix $\tilde{\boldsymbol{Z}}$ and vector $\boldsymbol{Y}$ without ith observation respectively, and $\tilde{w}_i = \tilde{\boldsymbol{z}}_i'\boldsymbol{B}^{-1}\tilde{\boldsymbol{z}}_i$ is ith diagonal element matrix $\tilde{\boldsymbol{W}} = \tilde{\boldsymbol{Z}}\boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'$ and $\tilde{e}_i(d) = y_i - \tilde{\boldsymbol{z}}_i'\hat{\boldsymbol{\theta}}_m(d)$ is ith residual obtained by fitting mixed Liu estimator for model (9). Also the inverse matrix of $\boldsymbol{B}$ is defined as follows

$$\boldsymbol{B}^{-1} = (\boldsymbol{\Pi} + \boldsymbol{I})^{-1}(\boldsymbol{I} + d\boldsymbol{\Pi}^{-1}) = \boldsymbol{G}_d\boldsymbol{\Pi}^{-1},$$

Then the weighted pseudo-values are given by

$$Q_i^* = \hat{\boldsymbol{\theta}}_m(d) + n(1 - \tilde{w}_i)(\hat{\boldsymbol{\theta}}_m(d) - \hat{\boldsymbol{\theta}}_{m(-i)}(d)). \tag{10}$$

Therefore, the Jackknifed Liu estimator under stochastic linear restrictions (6) for canonical model (9) is defined as follows

$$\tilde{\boldsymbol{\theta}}_m(d) = \frac{1}{n}\sum_{i=1}^{n} Q_i^*,$$

and due to $\sum_{i=1}^{n} \tilde{\boldsymbol{z}}_i y_i = \tilde{\boldsymbol{Z}}'\boldsymbol{Y}$ and $\sum_{i=1}^{n} \tilde{\boldsymbol{z}}_i \tilde{\boldsymbol{z}}_i' = \tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}}$, the mixed Jackknifed Liu estimator of $\boldsymbol{\theta}$ is as follows

$$\begin{aligned}\tilde{\boldsymbol{\theta}}_m(d) &= \hat{\boldsymbol{\theta}}_m(d) + \boldsymbol{B}^{-1}\sum_{i=1}^{n} \tilde{\boldsymbol{z}}_i \tilde{e}_i(d) \\ &= \hat{\boldsymbol{\theta}}_m(d) + \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\boldsymbol{Y} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}}\boldsymbol{B}^{-1}\boldsymbol{U}'\tilde{\boldsymbol{Y}} = (\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})\hat{\boldsymbol{\theta}}_m(d) + \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\boldsymbol{Y}.\end{aligned} \tag{11}$$

According to Equation (11), the bias vector of mixed Jackknifed Liu estimator is as follows

$$\begin{aligned}Bias(\tilde{\boldsymbol{\theta}}_m(D)) &= E(\tilde{\boldsymbol{\theta}}_m(D)) - \boldsymbol{\theta} = (\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})E(\hat{\boldsymbol{\theta}}_m(D)) + \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'E(\boldsymbol{Y}) - \boldsymbol{\theta} \\ &= (\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})\boldsymbol{G}_D\boldsymbol{\theta} + \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\boldsymbol{X}\boldsymbol{P}\boldsymbol{\theta} - \boldsymbol{\theta} \\ &= \left[(\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})\boldsymbol{G}_D + \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}} - \boldsymbol{I}\right]\boldsymbol{\theta} \\ &= (\boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}} - \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{G}_D)\boldsymbol{\theta} = -(\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})(\boldsymbol{I} - \boldsymbol{G}_D)\boldsymbol{\theta}.\end{aligned}$$

Also, the variance-covariance matrix of mixed Jackknifed Liu estimator is given as

$$\begin{aligned}Var\left[\tilde{\boldsymbol{\theta}}_m(d)\right] =&\sigma^2(\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})\boldsymbol{G}_d\boldsymbol{\Pi}^{-1}\boldsymbol{G}_d'(\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})' \\ &+ \sigma^2(\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})\boldsymbol{G}_d\boldsymbol{\Pi}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}}\boldsymbol{B}^{-1} \\ &+ \sigma^2\boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}}\boldsymbol{\Pi}^{-1}\boldsymbol{G}_d'(\boldsymbol{I} - \boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}})' \\ &+ \sigma^2\boldsymbol{B}^{-1}\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{Z}}\boldsymbol{B}^{-1}.\end{aligned}$$

Since $\boldsymbol{\theta} = \boldsymbol{P}'\boldsymbol{\beta}$, therefore the mixed, mixed Liu and mixed Jackknifed Liu estimators for Model (1) are $\hat{\boldsymbol{\beta}}_m = \boldsymbol{P}'\hat{\boldsymbol{\theta}}_m$, $\hat{\boldsymbol{\beta}}_m(d) = \boldsymbol{P}'\hat{\boldsymbol{\theta}}_m(d)$ and $\tilde{\boldsymbol{\beta}}_m(d) = \boldsymbol{P}'\tilde{\boldsymbol{\theta}}_m(d)$ respectively.

The MSE of new estimators has complicated forms, therefore an explicit expression for the estimator of d cannot be achieved. This would be possible through numerical methods which we will perform these methods in a simulation study and real example to obtain an optimal estimator of d.

5 Simulation study

In this section, we use a simulation study to evaluate the performance of the mixed Jackknifed Liu, ordinary least squares, mixed, Liu, mixed Liu and Jackknifed Liu under two *MSEM* and *ABIAS* criteria. In this study, following McDonald and Galarneau

(1975) and Kibria (2003) approach, we simulate the regression model in the following way. The data for the explanatory variables are generated as follows

$$x_{ij} = (1-\rho^2)^{1/2} w_{ij} + \rho w_{i,p+1}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, p, \tag{12}$$

where w_{ij}'s are independent standard normal pseudo-random numbers and ρ is determined so that the correlation between any two explanatory variables is given by ρ^2. In this study, we consider three different values for ρ 0.8, 0.9 and 0.99 to compare the results under weak, moderate, and severe multicollinearity. Following Newhouse and Oman (1971), McDonald and Galarneau (1975), Kibria (2003), and many other researchers we choose $\boldsymbol{\beta}$ as the normalized eigenvector corresponding to the smallest eigenvalue of $\boldsymbol{X}'\boldsymbol{X}$ matrix in this simulation study. Also, it should be noted that the model is generated without an intercept term. Then the observations on the response variable are generated as follows

$$y_i = \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \varepsilon_i, \qquad i = 1, 2, \ldots, n,$$

where ε_i's are independent random numbers from a normal distribution with mean 0 and σ^2 variance. Also in this study, the constraint matrix elements are derived from the $U(0,3)$ distribution and the number of constraints is assumed to be $m = 2$. The constraint vector is also produced as follows

$$r_i = \beta_1 R_{i1} + \ldots + \beta_p R_{ip} + \phi_i, \qquad i = 1, 2$$

For the σ parameter, three different values equal to 1, 4 and 9 are considered. We choose two values 30 and 100 for n and $p = 3$. Then the experiment is replicated 1000 times by generating new error terms.

Firinguetti (1989) compared different estimators under the criteria of MSE and absolute bias. Namura (1988) compared the ridge and the jackknifed ridge estimators under the criteria of MSE and biases by doing a simulation study. We will also use these criteria to investigate the performance of OLS, Liu and jackknifed Liu estimators. The MSE and the absolute bias for any estimator b are calculated respectively as follows

$$MSE\,(\boldsymbol{b}) = \frac{1}{1000} \sum_{r=1}^{1000} (\boldsymbol{b}_r - \boldsymbol{\beta})' (\boldsymbol{b}_r - \boldsymbol{\beta}),$$

$$ABIAS\,(\boldsymbol{b}) = \sum_{j=1}^{p} \left| \bar{b}_j - \beta_j \right|, \qquad \bar{b}_j = \frac{1}{1000} \sum_{r=1}^{1000} b_{j(r)},$$

where $b_{j(r)}$ is the estimator of β_j in the r-th replication of the experiment. For each replication, the value of d is estimated using relation (5) which minimizes the MSE of the Liu estimator. Statistical software R is used to perform this simulation. The numerical results of simulation study are presented in Tables 1-3.

The results show that in most cases the mixed Jackknifed Liu estimator has the best performance among all estimators regarding MSE and bias criteria. Therefore it can be stated that using Jackknifed technique and stochastic linear restrictions simultaneously leads to improvement of Liu estimator performance. When the sample size increases, convergence of the MSE to zero is a sufficient condition for consistency. We could easily check the consistency of these estimators.

Table 1: MSEs and absolute biases of OLS, OME, Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu estimators with $\sigma = 1$

		$\rho = 0.8$		$\rho = 0.9$		$\rho = 0.99$	
n	Estimator	MSE	$ABIAS$	MSE	$ABIAS$	MSE	$ABIAS$
$n = 30$	$\hat{\boldsymbol{\beta}}$	0.22442	0.59797	0.41724	0.83408	3.87715	1.73992
	$\hat{\boldsymbol{\beta}}_m$	0.17626	0.40910	0.30152	0.55358	1.63663	0.74758
	$\hat{\boldsymbol{\beta}}(d)$	0.22144	0.57910	0.40350	0.78381	3.02490	1.24432
	$\hat{\boldsymbol{\beta}}_m(d)$	0.18531	0.43512	0.31177	0.57889	1.41205	0.57136
	$\tilde{\boldsymbol{\beta}}(d)$	0.20875	0.40742	0.37294	0.51209	3.04974	0.60474
	$\tilde{\boldsymbol{\beta}}_m(d)$	0.16445	0.28128	0.27041	0.34480	1.21992	0.12600
$n = 100$	$\hat{\boldsymbol{\beta}}$	0.04993	0.48763	0.12127	0.00237	1.01684	0.55910
	$\hat{\boldsymbol{\beta}}_m$	0.04907	0.48482	0.11537	0.00377	0.62557	0.39534
	$\hat{\boldsymbol{\beta}}(d)$	0.04988	0.48632	0.12125	0.00244	0.95335	0.50018
	$\hat{\boldsymbol{\beta}}_m(d)$	0.04981	0.48566	0.12031	0.00302	0.62748	0.38308
	$\tilde{\boldsymbol{\beta}}(d)$	0.04890	0.43552	0.11930	0.01504	0.77679	0.24464
	$\tilde{\boldsymbol{\beta}}_m(d)$	0.04808	0.43326	0.11358	0.01604	0.50559	0.15398

Table 2: MSEs and absolute biases of OLS, OME, Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu estimators with $\sigma = 4$

		$\rho = 0.8$		$\rho = 0.9$		$\rho = 0.99$	
n	Estimator	MSE	$ABIAS$	MSE	$ABIAS$	MSE	$ABIAS$
$n = 30$	$\hat{\boldsymbol{\beta}}$	3.09942	5.16720	6.08547	5.01170	60.97988	3.16755
	$\hat{\boldsymbol{\beta}}_m$	2.52378	3.64221	4.55291	3.35178	26.24409	1.29889
	$\hat{\boldsymbol{\beta}}(d)$	3.04723	5.08765	5.83374	4.82724	46.58424	2.36328
	$\hat{\boldsymbol{\beta}}_m(d)$	2.63694	3.97840	4.64848	3.65700	21.58811	1.12910
	$\tilde{\boldsymbol{\beta}}(d)$	2.85391	4.45495	5.35741	3.95506	47.62215	1.40116
	$\tilde{\boldsymbol{\beta}}_m(d)$	2.29181	3.22502	3.90152	2.76388	17.79188	0.47815
$n = 100$	$\hat{\boldsymbol{\beta}}$	1.01643	0.80725	1.72959	0.99085	15.97173	1.43947
	$\hat{\boldsymbol{\beta}}_m$	0.98403	0.74872	1.61812	0.89742	10.81092	0.89934
	$\hat{\boldsymbol{\beta}}(d)$	1.01592	0.80667	1.72643	0.98862	14.58345	1.32308
	$\hat{\boldsymbol{\beta}}_m(d)$	1.01051	0.79992	1.69008	0.96866	10.69500	1.01514
	$\tilde{\boldsymbol{\beta}}(d)$	0.98906	0.77351	1.64043	0.92228	11.38428	0.90039
	$\tilde{\boldsymbol{\beta}}_m(d)$	0.95842	0.71708	1.54020	0.83663	8.13662	0.53906

6 Real example

To illustrate the performance of the new estimator we aim a data set on the productivity of electrical industrial power plants with 10 or more employees in Iran in 2006. The data were collected by the Iranian Statistics Center. First, the multicollinearity diagnostics are implemented and then we will fit a model to data using least squares, Liu and Liu Jackknifed estimators under two conditions: without restrictions and with restrictions. Then we compare these estimators based on the MSE and ABIAS criteria.

In this study, the response variable is labor productivity, which is calculated by dividing the value added by the number of employees in the plant, and the explanatory variables in this example are six variables as follows (it should be noted that the costs used in some explanatory variables are in million Rials):

Table 3: MSEs and absolute biases of OLS, OME, Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu estimators with $\sigma = 9$

n	Estimator	$\rho = 0.8$		$\rho = 0.9$		$\rho = 0.99$	
		MSE	$ABIAS$	MSE	$ABIAS$	MSE	$ABIAS$
$n = 30$	$\hat{\boldsymbol{\beta}}$	15.22895	8.66537	30.81025	6.48025	307.78840	4.14853
	$\hat{\boldsymbol{\beta}}_m$	12.60267	5.96462	23.24754	4.23711	132.66653	1.34063
	$\hat{\boldsymbol{\beta}}(d)$	14.96522	8.53981	29.51341	6.25722	234.07322	3.12629
	$\hat{\boldsymbol{\beta}}_m(d)$	13.15430	6.66719	23.70143	4.63010	108.46096	1.29741
	$\tilde{\boldsymbol{\beta}}(d)$	14.00343	7.55772	27.08790	5.19032	240.13840	1.92202
	$\tilde{\boldsymbol{\beta}}_m(d)$	11.37263	5.34194	19.78377	3.54948	89.35971	0.56376
$n = 100$	$\hat{\boldsymbol{\beta}}$	4.46421	1.54993	10.45353	1.42032	92.81526	2.00324
	$\hat{\boldsymbol{\beta}}_m$	4.36549	1.54163	10.18470	1.37968	56.54471	1.33488
	$\hat{\boldsymbol{\beta}}(d)$	4.46178	1.54910	10.42263	1.41643	83.72409	1.84542
	$\hat{\boldsymbol{\beta}}_m(d)$	4.45207	1.54786	10.36083	1.40964	53.91592	1.40368
	$\tilde{\boldsymbol{\beta}}(d)$	4.33277	1.50305	9.74966	1.32280	65.04305	1.33118
	$\tilde{\boldsymbol{\beta}}_m(d)$	4.23929	1.49585	9.51549	1.28623	42.11021	0.96765

Table 4: MSEs of propesed estimators

	n	$\hat{\boldsymbol{\beta}}_m(d)$			$\tilde{\boldsymbol{\beta}}_m(d)$		
		500	1000	100000	500	1000	100000
$\sigma = 2$	$\rho = 0.8$	0.05123	0.00229	0.00025	0.05037	0.00229	0.00025
	$\rho = 0.9$	0.08367	0.00439	0.00043	0.08307	0.00439	0.00043
	$\rho = 0.99$	0.82943	0.04098	0.00414	0.82998	0.04025	0.00413
$\sigma = 4$	$\rho = 0.8$	0.21942	0.00990	0.00099	0.19651	0.00984	0.00099
	$\rho = 0.9$	0.41637	0.01707	0.00173	0.34905	0.01684	0.00173
	$\rho = 0.99$	4.35398	0.17779	0.01592	3.13193	0.15921	0.01568
$\sigma = 9$	$\rho = 0.8$	1.38320	0.05150	0.00477	0.96250	0.04956	0.00475
	$\rho = 0.9$	2.83115	0.09587	0.00908	1.86247	0.08901	0.00900
	$\rho = 0.99$	25.9710	1.12442	0.09328	120.026	0.73066	0.08697

1. Number of employees in each plant;
2. Labor Costs (Employee Service Compensation);
3. Total inventory of capital assets at the end of the year;
4. Cost of raw materials;
5. Cost of energy consumed;
6. Costs related to other payments.

Information on the above variables was collected from 442 industrial plants. It should also be noted that the data on the productivity of electrical industry plants in 2005 is used as prior information in the form of stochastic linear restrictions. This information has been compiled and calculated through the industry section of the Statistical Yearbook (2005) and is as increment

$$r = 12.24, \quad \boldsymbol{R} = (65.49, 2.65, 4679.06, 9.72, 1240.16, 1583.75),$$

where r and $\boldsymbol{R}$ are defined according to (6). It should be noted that the values of r and the $\boldsymbol{R}$ are the average of the variables considered for the electrical industry plants in 2005. After fitting the various estimators, we compare the estimators under MSE and $ABIAS$ criteria. For d we consider a numerical sequence of 0 to 1 with a distance of 0.05. The results are presented in Tables 5 and 6.

Table 5: MSEs of Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu for different values of d

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\hat{\boldsymbol{\beta}}(d)$	2.5023	2.2592	2.0289	1.8115	1.6070	1.4154	1.2366	1.0707	0.9176	0.7775
$\hat{\boldsymbol{\beta}}_m(d)$	1.2827	1.1584	1.0408	0.9299	0.8256	0.7280	0.6371	0.5529	0.4753	0.4044
$\tilde{\boldsymbol{\beta}}(d)$	2.2379	1.8257	1.4751	1.1796	0.9333	0.7303	0.5655	0.4337	0.3304	0.2513
$\tilde{\boldsymbol{\beta}}_m(d)$	1.0893	0.9205	0.7732	0.6455	0.5355	0.4416	0.3623	0.2959	0.2410	0.1963
d	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
$\hat{\boldsymbol{\beta}}(d)$	0.6502	0.5357	0.4342	0.3455	0.2697	0.2067	0.1566	0.1194	0.0951	0.0836
$\hat{\boldsymbol{\beta}}_m(d)$	0.3402	0.2827	0.2318	0.1876	0.1500	0.1192	0.0930	0.0774	0.0666	0.0624
$\tilde{\boldsymbol{\beta}}(d)$	0.1922	0.1498	0.1205	0.1017	0.0905	0.0849	0.0829	0.0829	0.0838	0.0847
$\tilde{\boldsymbol{\beta}}_m(d)$	0.1605	0.1325	0.1110	0.0951	0.0838	0.0763	0.0718	0.0697	0.0692	0.0699

According to the results of Table 5, for d in the range of 0-0.85, the mixed Jackknifed Liu (the proposed estimator of this study) has the minimum MSE among other estimators. But for the values of d equals 0.9, 0.95, and 1 mixed Liu estimator has the best performance according to MSEs in this data set. In general, it can be said that increasing the value of d in most cases improves the performance of estimators according to the MSE criterion. In addition, for all values considered of d, the MSEs of estimators under restrictions are smaller than that of unrestricted estimators. So this could be another proof of the applicability of linear restrictions in reducing the effects of multicollinearity.

Table 6: Absolute biases of Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu for different values of d

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\hat{\boldsymbol{\beta}}(d)$	2.7773	2.6385	2.4996	2.3608	2.2219	2.0830	1.9441	1.8053	1.6664	1.5276
$\hat{\boldsymbol{\beta}}_m(d)$	1.9935	1.8938	1.7941	1.6944	1.5948	1.4951	1.3954	1.2958	1.1961	1.0964
$\tilde{\boldsymbol{\beta}}(d)$	2.5989	2.3455	2.1051	1.8777	1.6633	1.4619	1.2734	1.0980	0.9356	0.7862
$\tilde{\boldsymbol{\beta}}_m(d)$	1.7865	1.6389	1.4974	1.3643	1.2392	1.1200	1.0063	0.8981	0.7955	0.6985
d	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
$\hat{\boldsymbol{\beta}}(d)$	1.3887	1.2498	1.1109	0.9721	0.8332	0.6943	0.5555	0.4166	0.2777	0.1389
$\hat{\boldsymbol{\beta}}_m(d)$	0.9967	0.8971	0.7974	0.6977	0.5980	0.4984	0.3987	0.2990	0.1993	0.0997
$\tilde{\boldsymbol{\beta}}(d)$	0.6497	0.5263	0.4158	0.3184	0.2339	0.1624	0.1039	0.0585	0.0260	0.0065
$\tilde{\boldsymbol{\beta}}_m(d)$	0.6071	0.5213	0.4410	0.3663	0.2973	0.2338	0.1758	0.1244	0.0780	0.0365

Finally, the results of Table 6 show that for d in the range 0-0.55 the mixed Jackknifed Liu estimator has smaller $ABIAS$ than all other estimators. But for d equal to 0.6-1 the Jackknifed Liu estimator is better than other estimators under the absolute bias criterion. We summarized the comparison of Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu estimators under criteria MSEs in Figure 1. From Figure 1 it

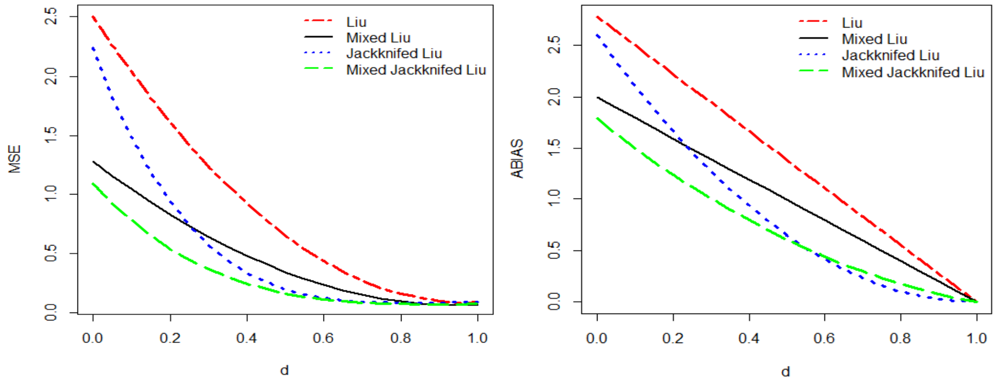

Figure 1: MSEs (left) and Absolute biases (right) of Liu, mixed Liu, Jackknifed Liu and mixed Jackknifed Liu for $0 < d < 1$

can be seen that the MSE of mixed Jackknifed Liu estimator is below the three other graphs, which the proposed estimator is more efficient compared to other estimators. Also, Figure 1 shows that for $0 < d < 0.55$ the mixed Jackknifed Liu estimator and for $0.55 < d < 1$ the Jackknifed Liu estimator have smaller $ABIAS$ among other estimators.

7 Conclusions

In this article, we proposed an alternative stochastic restricted Liu estimator for the vector of parameters in a linear regression model when additional stochastic linear restrictions on the parameter vector are assumed to hold. Then, the Jackknife technique was performed to reduce the bias of the new estimator and we obtained another new estimator which we called the mixed Jackknifed Liu estimator. Since the new estimator had a complex form, using a simulation study its performance was investigated under MSEs and absolute bias criteria. Simulation results have shown that the new stochastic restricted Jackknifed Liu estimator outperforms the Liu, stochastic restricted Liu, and Jackknifed Liu estimators in the MSE and $ABIAS$ senses. Also, data set on the productivity of industrial power plants were used and the results showed that the proposed estimator has smaller MSEs compare with other estimators. It was also observed that the application of the Jackknife technique significantly reduces the absolute biases. We have established that using both Jackknifed and linear restrictions techniques improves the efficiency of the resulting estimator. The results showed that, for some values of d, the mixed Jackknifed Liu estimator has the best performance concerning MSE and $ABIAS$ criteria. But what can be concluded from simulation and real example results is that imposing restrictions on parameter vector reduces the MSE of the corresponding estimator. Also, in most cases, the $ABIAS$ criterion has decreased. On the other hand, the Jackknifed technique has a good effect on reducing the $ABIAS$ of all discussed estimators.

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