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Research Paper

The exponentiated half logistic-log-logistic Weibull distribution: Model, properties and applications

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Abstract: We develop a generalized distribution, namely, exponentiated half logistic log-logistic Weibull distribution. Several structural properties of the distribution including expansion of density, distribution of order statistics, Rényi entropy, moments, probability weighted moments, quantile function, generating function, and maximum likelihood estimates were derived. A simulation study to examine the consistency of the maximum likelihood estimates was conducted. Finally, real data examples are presented to illustrate the applicability and usefulness of the proposed model.

Keywords: Generalized distribution; Half logistic distribution; Log-logistic distribution; Maximum likelihood estimation; Weibull distribution.
Mathematics Subject Classification (2010): 60E05, 62E15, 62F30.

1 Introduction

Increasing demand for generalized distributions in areas of finance, economics, and physics just to mention a few, has motivated statisticians to work on improving classical models by adding some extra parameters to these models. The log-logistic and Weibull distributions are widely used in many areas of biology, reliability, and insurance. However, these distributions have an inadequate range of behavior and failed to provide adequate fit in some real lifetime situations. Therefore, new classes of distributions originated from modified versions of the Weibull distribution were proposed to fulfill the non-monotonic failure rate. Some of these generalized and extended distributions include the beta modified Weibull (BMW) distribution by Nadarajah et al.

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(2011), the gamma generalized modified Weibull (GGMW) distribution by Oluyede et al. (2015), gamma Weibull-G family by Oluyede et al. (2018), the exponentiated Weibull (EW) distribution by Gupta et al. (2001) and the modified Weibull (MW) distribution by Lai et al. (2003), to mention a few.

The half-logistic distribution plays an important role in many practical situations such as physics and hydrology. The distribution was developed by Balakrishman et al. (1985). Extensions of the half-logistic distribution include the odd exponentiated half-logistic-G distribution by Afify et al. (2017) and the type 1 half-logistic distribution by Cordeiro et al. (2016). In this work, we study a generalized distribution called the exponentiated half-logistic log-logistic Weibull (EHL-LLoGW) distribution.

The exponentiated half-logistic distribution developed by Cordeiro et al. (2014), has cumulative distribution function (cdf) and probability density function (pdf) given by

$$F(x;\phi) = \left[\frac{1-\bar{G}(x,\phi)}{1+\bar{G}(x,\phi)}\right]^{\delta}, \tag{1}$$

$$f(x;\phi) = \frac{2\delta g(x,\phi)(1-\bar{G}(x,\phi))^{\delta-1}}{(1+\bar{G}(x,\phi))^{\delta+1}}. \tag{2}$$

respectively, where $G(x;\phi)$ is the baseline cdf, $\bar{G}(x;\phi) = 1-G(x;\phi)$ and ϕ is the vector of parameters from the baseline distribution.

Oluyede et al. (2016) developed the log-logistic Weibull (LLoGW) distribution with cdf and pdf given by

$$G_{LLoGW}(x) = 1-\left(1+x^{c}\right)^{-1}\exp(-\alpha x^{\beta}), \tag{3}$$

$$g_{LLoGW}(x) = e^{-\alpha x^{\beta}}\left(1+x^{c}\right)^{-1}\left\{\alpha x^{\beta-1}+\frac{cx^{c-1}}{(1+x^{c})}\right\}, \tag{4}$$

respectively, for $c, \alpha, \beta > 0$ and $x \geq 0$.

The main motivation for the development of the exponentiated half logistic-log logistic Weibull (EHL-LLoGW) distribution is the applicability of the log-logistic and Weibull distributions in different areas of science and engineering, the ability to handle skewness and kurtosis, ability to handle monotonic and non-monotonic hazard rates. The proposed distribution is versatile and flexible in data fitting. The rest of the paper is organized as follows: In Section 2, we present the generalized distribution and the expansion of its density. Some sub-models, hazard, and reverse hazard rate functions of the EHL-LLoGW distribution are also presented in this section. Structural properties including the distribution of order statistics, Rényi entropy, moments, probability weighted moments, quantile and generating functions are presented in Section 3. In Section 4, we present the maximum likelihood estimates. The Monte Carlo simulation study is conducted in Section 5. Applicability of the EHL-LLoGW distribution is tested in Section 6, followed by concluding remarks.

2 The proposed distribution

2.1 Model definition

In this section, we derive the EHL-LLoGW distribution and obtain a series expansion of the distribution. We also include sub-models, hazard and reverse hazard rate functions as well as the graphs of the pdf and hazard rate function (hrf). We transform the LLoGW distribution using the half logistic generator and add an exponentiation parameter to obtain the EHL-LLoGW distribution with cdf and pdf given by

$$
\begin{aligned}
F(x;\alpha,\beta,\delta,c) &= \left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)^{\delta}, &(5)\\
f(x;\alpha,\beta,\delta,c) &= \frac{2\delta e^{-\alpha x^{\beta}}(1+x^c)^{-1}\left[\alpha x^{\beta-1}+\frac{cx^{c-1}}{(1+x^c)}\right]}{\left[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}\right]^2}\\
&\quad\times\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)^{\delta-1}, &(6)
\end{aligned}
$$

respectively, for $c,\alpha,\beta>0,\delta>0$ and $x\geq 0$.

2.2 Series expansion

We present the series expansion of the EHL-LLoGW distribution in this section. By applying the following generalized binomial series expansions

$$
\begin{aligned}
[1+\bar{G}_{LLoGW}(x;\alpha,\beta,c)]^{-(\delta+1)} &= \sum_{p=0}^{\infty}(-1)^p\binom{-(\delta+1)}{p}\bar{G}^p_{LLoGW}(x;\alpha,\beta,c),\\
\left[1-\bar{G}_{LLoGW}(x;\alpha,\beta,c)\right]^{\delta-1} &= \sum_{m=0}^{\infty}(-1)^m\binom{\delta-1}{m}\bar{G}^m_{LLoGW}(x;\alpha,\beta,c),\\
\left[\bar{G}_{LLoGW}(x;\alpha,\beta,c)\right]^{p+m} &= (1-G_{LLoGW}(x;\alpha,\beta,c))^{p+m}\\
&= \sum_{j=0}^{\infty}(-1)^j\binom{p+m}{j}G^j_{LLoGW}(x;\alpha,\beta,c),
\end{aligned}
$$

we can write the pdf of the EHL-LLoGW distribution as

$$
\begin{aligned}
f(x;\alpha,\beta,c) &= \sum_{p,m,j=0}^{\infty}2\delta(-1)^{p+m+j}\binom{-(\delta+1)}{p}\binom{\delta-1}{m}\binom{p+m}{j}\\
&\quad\times g_{LLoGW}(x;\alpha,\delta,c)G^j_{LLoGW}(x;\alpha,\beta,c)\\
&= \sum_{j=0}^{\infty}v_j g_j(x;\alpha,\beta,c), &(7)
\end{aligned}
$$

which is an infinite linear combination of exponentiated-LLoGW (Exp-LLoGW) distribution, where

$$g_j(x;\alpha,\beta,c) = (j+1)g_{LLoGW}(x;\alpha,\beta,c)[G^j_{LLoGW}(x;\alpha,\beta,c)],$$

is an Exp-LLoGW distribution with power parameter j and linear component

$$v_j = \sum_{p,m=0}^{\infty} \frac{2\delta(-1)^{p+m+j}}{(j+1)} \binom{-(\delta+1)}{p}\binom{\delta-1}{m}\binom{p+m}{j}. \tag{8}$$

We can therefore, derive other mathematical properties of the EHL-LLoGW distribution directly from those of the Exp-LLoGW distribution.

2.3 Some sub-models of ELLLoGW distribution

In this sub-section, we discuss some sub-models of EHL-LLoGW distribution.

- If $\delta = 1$, we obtain the half logistic log-logistic Weibull (HL-LLoGW) distribution.
- If $\beta = 1$, we obtain the exponentiated half logistic-log-logistic exponential (EHL-LLoGE) distribution.
- If $\beta = 2$, we obtain the exponentiated half logistic-log-logistic Rayleigh (EHL-LLoGR) distribution.
- If $\delta = \beta = 1$, we obtain the half logistic-log logistic Exponential (HL-LLoGE) distribution.
- If $\alpha = 0$, we obtain the exponentiated half logistic-log-logistic model (EHL-LLoG) distribution.
- If δ = c = 1, the EHL-LLoGW cdf reduces to the two parameter distribution with cdf given by

$$F(x;\alpha,\beta) = \left(\frac{1-(1+x)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x)^{-1}e^{-\alpha x^{\beta}}}\right) \tag{9}$$

  for $\alpha, \beta > 0$ and $x \geq 0$.

2.4 Hazard and reverse hazard rate functions

We present the hazard and reverse hazard rate functions of the EHL-LLoGW distribution in this section. The hazard and reverse hazard functions of the EHL-LLoGW distribution are respectively given by

$$\begin{aligned} h_F(x) &= 2e^{-\alpha x^{\beta}}(1+x^c)^{-1}\left[\alpha x^{\beta-1} + \frac{cx^{c-1}}{(1+x^c)}\right] \\ &\quad \times \left[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}\right]^{-2}\left(1-\left[\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right]\right)^{-1}, \end{aligned}$$

$$\tau_F(x) = 2e^{-\alpha x^{\beta}}(1+x^c)^{-1}\left[\alpha x^{\beta-1} + \frac{cx^{c-1}}{(1+x^c)}\right] \times \left[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}\right]^{-2}\left[\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right]^{-1},$$

for $x \geq 0$, $c, \alpha, \beta, \delta > 0$.

Figure 1 shows the plots of the pdfs and hrfs of EHL-LLoGW distribution for selected parameters values. The pdf can take various shapes including almost symmetric, right and left-skewed, J and reverse-J. Graphs of the hrf exhibits increasing, decreasing, reverse-J and uni-modal shapes.

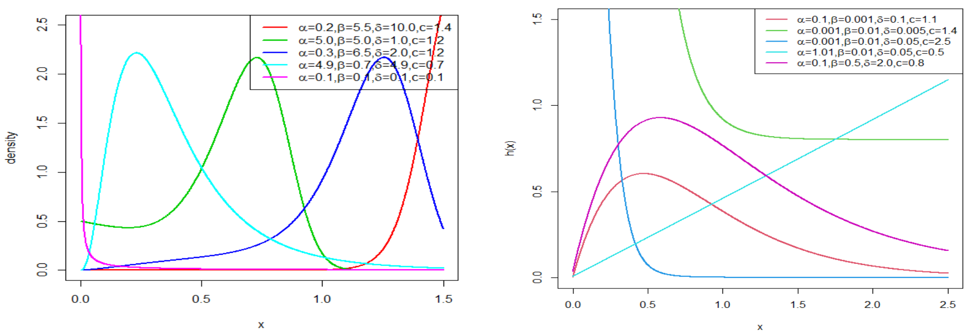

Figure 1: Plots of the pdf and hrf for the EHL-LLoGW distribution.

3 Some properties

In this section, we derive some properties of the EHL-LLoGW distribution which includes distribution of order statistics, Rényi entropy, ordinary and incomplete moments, probability weighted moments (PWMs), generating function and quantile function.

3.1 Distribution of order statistics

Order statistics plays a vital role in many areas of statistical theory and practice. Let $X_1, ..., X_n$ be a random sample from the EHL-LLoGW distribution. The pdf of the i^{th} order statistic can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-j} \binom{n-i}{j} F(x)^{j+i-1}, \tag{10}$$

where $B(.,.)$ is the beta function. Substituting equations of the cdf and pdf of the EHL-LLoGW into equation (10), we have

$$f(x)F(x)^{j+i-1} = \frac{2\delta g_{LLoGW}(x;\alpha,\beta,c)(1-\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\delta-1}}{(1+\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\delta+1}}$$

$$
\begin{aligned}
& \times\left(\left[\frac{1-\bar{G}_{LLoGW}(x;\alpha,\beta,c)}{1+\bar{G}_{LLoGW}(x;\alpha,\beta,c)}\right]^{\delta}\right)^{j+i-1} \\
= & \quad \frac{2\delta g_{LLoGW}(x;\alpha,\beta,c)(1-\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\delta(j+i)-1}}{(1+\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\delta(j+i)+1}}.
\end{aligned}
$$

Using the following generalized binomial series expansions

$$
\begin{aligned}
(1+\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{-(\delta(j+i)+1)} &= \sum_{z=0}^{\infty}(-1)^{z}\binom{-(\delta(j+i)+1)}{z} \\
&\quad \times\bar{G}^{z}_{LLoGW}(x;\alpha,\beta,c), \\
(1-\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\delta(j+i)-1} &= \sum_{k=0}^{\infty}(-1)^{k}\binom{\delta(j+i)-1}{k} \\
&\quad \times\bar{G}^{k}_{LLoGW}(x;\alpha,\beta,c), \\
\overline{G}^{z+k}_{LLoGW}(x;\alpha,\beta,c) &= (1-G_{LLoGW}(x;\alpha,\beta,c))^{z+k} \\
&= \sum_{m=0}^{\infty}(-1)^{m}\binom{z+k}{m}G^{m}_{LLoGW}(x;\alpha,\beta,c),
\end{aligned}
$$

we have

$$
\begin{aligned}
f(x)F(x)^{j+i-1} &= \sum_{z,k,m=0}^{\infty}(-1)^{m+z+k}2\delta\binom{-\delta(j+i)+1}{z}\binom{\delta(j+i)-1}{k}\binom{z+k}{m} \\
&\quad \times g_{LLoGW}(x;\alpha,\beta,c)G^{m}_{LLoGW}(x;\alpha,\beta,c).
\end{aligned} \tag{11}
$$

We can therefore, write pdf of the i^{th} order statistic of the EHL-LLoGW distribution as

$$
\begin{aligned}
f_{i:n}(x) &= \frac{1}{\mathbf{B}(i,n-i+1)}\sum_{z,k,m=0}^{\infty}\sum_{j=0}^{n-j}(-1)^{m+z+k}2\delta\binom{-\delta(j+i)+1}{z}\binom{\delta(j+i)-1}{k} \\
&\quad \times\binom{z+k}{m}g_{LLoGW}(x;\alpha,\beta,c)(G_{LLoGMW}(x;\alpha,\beta,c))^{m} \\
&= \sum_{m=0}^{\infty}v_m h_m(x;\alpha,\beta,c),
\end{aligned} \tag{12}
$$

where $\mathbf{B}(.,.)$ is the beta function,

$$
h_m(x) = (m+1)g_{LLoGW}(x;\alpha,\beta c)(G_{LLoGMW}(x;\alpha,\beta,c))^{m}
$$

is the Exp-LLoGW density with power parameter m and the weights v_m are given by

$$
v_m = \frac{1}{\mathbf{B}(i,n-i+1)}\sum_{z,k=0}^{\infty}\sum_{j=0}^{n-j}\frac{(-1)^{m+k+z}2\delta}{(m+1)}\binom{-\delta(j+i)+1}{z}
$$

$$\times\binom{\delta(j+i)-1}{k}\binom{z+k}{m}. \tag{13}$$

Then, the density function from the i^{th} order statistics from the EHL-LLoGW distribution is a mixture of Exp-LLoGW densities. We note that the mathematical and statistical properties of distribution of the i^{th} order statistic from the EHL-LLoGW distribution follow from those properties of Exp-LLoGW distribution.

3.1.1 Moments of order statistics

The q^{th} moment of the i^{th} order statistics from the EHL-LLoGW can be expressed as

$$E(X_{i:n}^q)=\sum_{m=0}^{\infty}v_m E(Y_m^q), \tag{14}$$

where Y_m follows an Exp-LLoGW distribution with power parameter m.

3.2 Entropy

There are several types of entropy, including Rényi entropy by Rényi et al. (1960) and Shannon entropy by Shannon et al. (1951). Shannon entropy is a special case of Rényi entropy. Rényi entropy ($I_R(\nu)$) is mathematically defined as

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\int_0^\infty f^\nu(x)dx\right], v\neq 1, v>0. \tag{15}$$

Using equation (6), $f^\nu(x)$, can be written as

$$f^\nu(x) = \frac{(2\delta)^\nu g^\nu_{LLoGW}(x;\alpha,\beta,c)(1-\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\nu(\delta-1)}}{(1+\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\nu(\delta+1)}}.$$

Appplying the following generalized binomial series expansions

$$\begin{aligned}
(1+\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{-\nu(\delta+1)} &= \sum_{j=0}^{\infty}(-1)^j\binom{-\nu(\delta+1)}{j}\overline{G}^j_{LLoGW}(x;\alpha,\beta,c),\\
(1-\bar{G}_{LLoGW}(x;\alpha,\beta,c))^{\nu(\delta+1)} &= \sum_{m=0}^{\infty}(-1)^m\binom{\nu(\delta+1)}{m}\overline{G}^m_{LLoGW}(x;\alpha,\beta,c),\\
\overline{G}^{m+j}_{LLoGW}(x;\alpha,\beta,c) &= (1-G_{LLoGW}(x;\alpha,\beta,c))^{m+j}\\
&= \sum_{p=0}^{\infty}(-1)^p\binom{m+j}{p}G^p_{LLoGW}(x;\alpha,\beta,c),
\end{aligned}$$

we have

$$\begin{aligned}
f^\nu(x) &= \sum_{j,m,p=0}^{\infty}(-1)^{j+m+p}(2\delta)^\nu\binom{-\nu(\delta+1)}{j}\binom{\nu(\delta+1)}{m}\binom{m+j}{p}\\
&\quad\times g^\nu_{LLoGW}(x;\alpha,\beta,\delta,c)G^p_{LLoGW}(x;\alpha,\beta,c).
\end{aligned} \tag{16}$$

Therefore, Rényi entropy of the EHL-LLoGW distributions can be written as

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\sum_{p=0}^{\infty} w_p e^{(1-\nu) I_{REG}} \right], v \neq 1, v > 0, \tag{17}$$

where

$$w_p = \sum_{j,m=0}^{\infty} (-1)^{j+m+p} (2\delta)^{\nu} \binom{-\nu(\delta+1)}{j} \binom{\nu(\delta+1)}{m} \binom{m+j}{p} \frac{1}{\left(\frac{p}{\nu}+1\right)^{\nu}}$$

and $I_{REG} = \int_0^\infty \left[\left(\frac{p}{\nu}+1\right) g_{LLoGW}(x;\alpha,\beta,c) [G_{LLoGW}(x;\alpha,\beta,c)]^{\frac{p}{\nu}} \right]^{\nu} dx$ is Rényi entropy of Exp-LLoGW distribution with parameter $\frac{p}{\nu}$. Therefore, Rényi entropy of the EHL-LLoGW distribution can be derived directly from Rényi entropy of Exp-LLoGW distribution.

3.3 Ordinary and incomplete moments

Since the EHL-LLoGW distribution can be expressed as an infinite linear combination of the Exp-LLoGW distribution, we therefore use the properties of the Exp-LLoGW distribution to derive the moments and generating function of the EHL-LLoGW distribution. The r^{th} ordinary moment of the EHL-LLoGW distribution is given by

$$\mu_r' = E(X^r) = \sum_{j=0}^{\infty} v_j E(Y_j^r), \tag{18}$$

where v_j is the linear component and is given by equation (8) and $E(Y_j^r)$ is the r^{th} moment of the Exp-LLoGW distribution. Also, using results from the Exp-LLoGW distribution, the r^{th} incomplete moment of X is given by

$$\phi_r(h) = \int_{-\infty}^{h} x^r f(x) dx = \sum_{j=0}^{\infty} v_j \int_{-\infty}^{h} x^r g_j(x;\alpha,\beta,c) dx, \tag{19}$$

where $\int_{-\infty}^{h} x^r g_j(x;\alpha,\beta,c)dx$ is the r^{th} incomplete moment of the Exp-LLoGW distribution.

We present the first five moments together with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the EHL-LLoGW distribution for selected values of the model parameters (see Table 1, for details).

In addition, we can obtain the moment generating function (mgf) of the EHL-LLoGW distribution as follows

$$M_X(t) = E(e^{tX}) = \sum_{j=0}^{\infty} v_j M_{Y_j}(t),$$

where $M_{Y_j}(t)$ is the mgf of the Exp-LLoGW distribution with power parameter j.

Table 1: Moments of the EHL-LLoGW distribution for some parameter values

	(0.4,0.5,0.5,1)	(0.5,1.5,1,1.5)	(0.5,0.5,1,0.5)	(1,1.5,1,0.5)	(1.3,1.5,1.5,0.5)
E(X)	0.1663	0.2910	0.1461	0.2836	0.3412
E(X^2)	0.0911	0.1957	0.0818	0.1806	0.2285
E(X^3)	0.0620	0.1461	0.0564	0.1319	0.1704
E(X^4)	0.0468	0.1160	0.0430	0.1036	0.1353
E(X^5)	0.0376	0.0960	0.0347	0.0851	0.1118
SD	0.2518	0.3331	0.2459	0.3164	0.3348
CV	1.5140	1.1447	1.6835	1.1159	0.9810
CS	1.6135	0.6639	1.8050	0.7539	0.4261
CK	4.5728	1.9416	5.2338	2.1622	1.7194

3.4 Probability weighted moments

The PWM is the expectation of certain function of a random variable whose mean exists. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The $(j,i)^{th}$ PWM, say $\eta_{j,i}$ of X is defined by

$$\eta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j f(x) F(x)^i dx.$$

Using (11), we can write

$$\begin{aligned} f(x)F(x)^i &= \sum_{z,k,m=0}^{\infty} (-1)^{z+k} 2\delta \binom{-\delta(1+i)+1}{z} \binom{\delta(1+i)-1}{k} \binom{z+k}{m} \\ &\quad \times g_{LLoGW}(x;\alpha,\beta,c) G_{LLoGW}^{m}(x;\alpha,\beta,c). \end{aligned} \tag{20}$$

which can be expressed as $f(x)F(x)^i = \sum_{m=0}^{\infty} v_m^* g_m(x;\alpha,\beta,c)$, where

$$v_m^* = \sum_{z,k=0}^{\infty} (-1)^{z+k} 2\delta \binom{-\delta(1+i)+1}{z} \binom{\delta(1+i)-1}{k} \binom{z+k}{m}$$

and $g_m(x;\alpha,\beta,c)$ is an Exp-LLoGW density with power parameter m. Then, the PWM of X is given by

$$\eta_{j,i} = \sum_{m=0}^{\infty} v_m^* \int_{-\infty}^{\infty} x^j g_m(x;\alpha,\beta,c) dx = \sum_{m=0}^{\infty} v_m^* E(T_m^j),$$

where T_m^j is j^{th} power of an Exp-LLoGW distributed random variable with power parameter m.

3.5 Quantile function

The quantile function of the EHL-LLoGW distribution is obtained by inverting the cdf given by equation (5), as follows;

$$F(x;\alpha,\beta,\delta,c) = \left(\frac{1-(1+x^c)^{-1} e^{-\alpha x^\beta}}{1+(1+x^c)^{-1} e^{-\alpha x^\beta}} \right)^\delta = u,$$

for $0 \leq u \leq 1$, that is,

$$(1+x^c)^{-1}e^{-\alpha x^{\beta}} = \frac{1-u^{\frac{1}{\delta}}}{1+u^{\frac{1}{\delta}}}.$$

The equation simplifies to

$$\log(1+x^c)^{-1} - \alpha x^{\beta} = \log\left[\frac{1-u^{\frac{1}{\delta}}}{1+u^{\frac{1}{\delta}}}\right].$$

Therefore, the quantiles of the EHL-LLoGW distribution may be determined by solving the equation

$$-\log(1+x^c) - \alpha x^{\beta} - \log\left[\frac{1-u^{\frac{1}{\delta}}}{1+u^{\frac{1}{\delta}}}\right] = 0. \tag{21}$$

Consequently, random numbers of the EHL-LLoGW distribution can be generated by solving equation (21) using R or Matlab. Some quantiles for selected parameters values are given in Table 2.

Table 2: Table of quantiles for selected parameters values of the EHL-LLoGW distribution

u	(1.2,0.5,1.1,0.5)	(1,1,0.5,0.5)	(0.5,0.5,1.5,1)	(1,0.5,1,0.5)	(1,1,1,0.5)
0.1	0.0133	0.0004	0.2232	0.0106	0.0332
0.2	0.0503	0.0059	0.4554	0.0452	0.1143
0.3	0.1141	0.0274	0.7179	0.1104	0.2284
0.4	0.2121	0.0774	1.0313	0.2165	0.3714
0.5	0.3582	0.1676	1.4250	0.3815	0.5455
0.6	0.5781	0.3107	1.9499	0.6385	0.7595
0.7	0.9251	0.5265	2.7102	1.0562	1.0333
0.8	1.5336	0.8601	3.9723	1.8089	1.4138
0.9	2.9037	1.4616	6.7618	3.5530	2.0551

4 Maximum likelihood estimation

Let X $\sim$EHL-LLoGW (α,β,δ,c) and $\theta = (\alpha,\beta,\delta,c)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\theta)$ for a single observation x of X is given by

$$\begin{aligned}\ell(\theta) &= \log 2 - \alpha x^{\beta} - \log(1+x^c) + \log(\alpha x^{\beta-1} + cx^{c-1}(1+x^c)^{-1}) \\ &\quad -2\log(1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}) + (\delta-1)\log\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right). \end{aligned}\tag{22}$$

The score functions for the parameters α,β,δ and c are given by

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= -x^{\beta} + \frac{\beta x^{\beta-1}}{\alpha\beta x^{\beta-1} + cx^{c-1}(1+x^c)^{-1}} + \frac{2(1+x^c)^{-1}x^{\beta}e^{-\alpha x^{\beta}}}{[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}]} \\ &\quad \times \frac{(\delta-1)2x^{\beta}(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}]^2}\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)^{-1},\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{\alpha(x^{\beta-1}+\beta x^{\beta-1}\log(x))}{[\alpha\beta x^{\beta-1}+cx^{c-1}(1+x^c)^{-1}]}+\frac{2(1+x^c)^{-1}\alpha x^{\beta}\log(x)e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{\alpha x^{\beta}}}\\
&+\frac{(\delta-1)2\alpha x^{\beta}\log(x)(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}]^2}\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)^{-1}\\
&-\alpha x^{\beta}\log(x),\\
\frac{\partial \ell}{\partial \delta} &= \log\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)\\
\frac{\partial \ell}{\partial c} &= -\frac{x^c\log(x)}{(1+x^c)}+\frac{[x^{c-1}+cx^{c-1}\log(x)](1+x^c)-x^c\log(x)cx^{c-1}}{\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)+cx^{c-1}(1+x^c)^{-1}}\\
&+\frac{2x^c\log(x)(1+x^c)^{-2}e^{-\alpha x^{\beta}}}{[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}]}+\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)^{-1}\\
&\times\frac{(\delta-1)2x^c\log(x)(1+x^c)^{-2}e^{-\alpha x^{\beta}}}{[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}]^2}.
\end{aligned}$$

Given a random of size n drawn from the EHL-LLoGW distribution, the total log-likelihood function is given by $\ell_n^*(\theta)=\sum_{j=1}^{n}\ell_j(\theta)$, where $\ell_j(\theta), j=1,2,...,n$ is given by equation (22). The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters α,β,δ and c must be found by using iterative methods. Under conditions that are fulfilled for parameters in the interior of the parameter space but on the boundary, we have: $\sqrt{n}(\hat{\theta}-\theta)$ is $N_4(\mathbf{0},\mathbf{I^{-1}}(\theta))$, where $\mathbf{I}(\theta)$ is the expected Fisher information matrix. This asymptotic behavior is valid if $I(\theta)$ is replaced by $J(\hat{\theta})$, the observed information matrix evaluated at $\hat{\theta}$. The multivariate normal $N_4(\mathbf{0},J(\hat{\theta})^{-1})$, where the mean vector $\mathbf{0}=(0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for $\alpha,\beta,\delta,$ and c are given by:

$$\hat{\alpha}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\theta})},\hat{\beta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\theta})},\hat{\delta}\pm Z_{\frac{\eta}{2}}\sqrt{I_{\delta\delta}^{-1}(\hat{\theta})},\hat{c}\pm Z_{\frac{\eta}{2}}\sqrt{I_{cc}^{-1}(\hat{\theta})},$$

where $I_{\alpha\alpha}^{-1}(\hat{\theta}),I_{\beta\beta}^{-1}(\hat{\theta}),I_{\delta\delta}^{-1}(\hat{\theta})$, and $I_{cc}^{-1}(\hat{\theta})$ are the diagonal elements $I_n^{-1}(\hat{\theta})=(nI(\hat{\theta}))^{-1}$, and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ percentile of a standard normal distribution.

5 Simulation study

In this section, the performance of the EHL-LLoGW distribution is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400 and 800) via the R package. We simulate $N=1000$ samples for the true parameters values given in Table 3. The table lists the mean MLEs of the model parameters along with the respective bias and root mean squared errors (RMSEs). The precision of the MLEs is discussed by means of the following measures: mean, mean square error (MSE) and average bias.

The estimated parameter values in Table 3 indicate that the estimates are quite stable and, more importantly, are close to the true parameter values for these sample sizes. The simulation study shows that the maximum likelihood method is appropriate for estimating the EHL-LLoGW model parameters. In fact, the means of the parameters tend to be closer to the true parameter values when n increases.

The bias and RMSE for the estimated parameter, $\hat{\theta}$, respectively, are given by

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}}.$$

6 Applications

The applicability of the model compared to several other models is tested in this section. Computations of the estimates of the model parameters was done using the **nlm** function in **R** software. Tables 5, 7 and 9 lists the MLEs (and standard errors in parenthesis) of the model parameters and the values of the goodness-of-fit-statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von-Mises (W*), and Andersen-Darling (A*) as described by Chen et al. (1985). The distribution with the lowest values of AIC, AICC, BIC, W* and A* is regarded as the best fitting model. Kolmogorov-Smirnov (KS) and sum of squares (SS) were also used to assess goodness-of-fit. The model with the smaller KS value and the highest p-value for the KS statistic is deemed as the best model.

Furthermore, we present plots of the fitted densities, the histogram of the data and probability plots by Chambers et al. (1983) to show how versatile the EHL-LLoGW model is compared to the other models. We compare the EHL-LLoGW distribution to several models, some of which are extensions of the Weibull distribution. The models considered are the Beta-Weibull (BW) by Cordeiro et al. (2013), Kumaraswamy-Weibull (KwW) by Cordeiro et al. (2010), the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. (2018), Topp-Leone-Weibull-Lomax (TL-WLx) by Jamal et al. (2019), and beta odd Lindley- Uniform (BOL-U) by Cordeiro et al. (2019).

The pdfs of the models are as follows:

$$\begin{aligned}
f_{BW}(x;a,b,\alpha,\beta) &= \frac{\beta\alpha^{\beta}}{B(a,b)} x^{\beta-1} e^{-b(\alpha x)^{\beta}} (1-e^{-(\alpha x)^{\beta}})^{a-1}, \\
f_{KwW}(x;a,b,\alpha,\beta) &= ab\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}} (1-e^{-(\alpha x)^{\beta}})^{a-1} (1-(1-e^{-(\alpha x)^{\beta}})^{a})^{b-1}, \\
f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda) &= \frac{\alpha\theta^{2}\gamma\lambda^{\gamma}x^{\gamma-1}e^{-(\lambda x)^{\gamma}}(e^{-(\lambda x)})^{\alpha\theta-1}(1-e^{-(\lambda x)^{\gamma}})^{\alpha-1}}{(\theta+\beta)((1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}})^{\theta-1}} \\
&\quad \times \left(1-\beta\log\left[\frac{e^{-(\lambda x)^{\gamma}}}{(1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}}}\right]\right), \\
f_{TL-WLx}(x;a,b,\alpha,\theta) &= 2\theta\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1} \\
&\quad \times \exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right)
\end{aligned}$$

Table 3: Monte Carlo simulation results for EHL-LLoGW distribution: Mean, RMSE and average bias

	I	$\alpha=1.5,\beta=1.5,\delta=1.5,c=0.3$			$\alpha=0.5,\beta=1.5,\delta=1.5,c=0.5$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
α	50	2.366	0.909	0.366	0.408	0.377	-0.091
	100	2.211	0.667	0.211	0.449	0.282	-0.050
	200	2.092	0.489	0.092	0.448	0.213	-0.051
	400	2.044	0.370	0.044	0.469	0.156	-0.030
	800	2.011	0.278	0.011	0.485	0.107	-0.014
β	50	2.277	2.063	0.277	2.104	1.208	0.604
	100	2.085	0.911	0.085	1.800	0.728	0.300
	200	2.045	0.612	0.045	1.673	0.442	0.173
	400	2.002	0.387	0.002	1.570	0.243	0.078
	800	1.995	0.287	-0.004	1.536	0.150	0.036
δ	50	0.759	0.604	0.259	1.698	8.291	0.198
	100	0.646	0.378	0.146	1.486	0.500	-0.013
	200	0.573	0.265	0.073	1.460	0.358	-0.039
	400	0.539	0.187	0.039	1.471	0.260	-0.028
	800	0.513	0.131	0.013	1.484	0.178	-0.015
c	50	1.792	1.299	0.292	0.772	0.557	0.272
	100	1.761	1.085	0.261	0.652	0.410	0.152
	200	1.790	0.934	0.290	0.598	0.274	0.098
	400	1.709	0.744	0.209	0.557	0.184	0.057
	800	1.698	0.620	0.198	0.525	0.112	0.025
	II	$\alpha=1.0,\beta=1.5,\delta=1.5,c=0.3$			$\alpha=1.0,\beta=1.0,\delta=0.5,c=1.5$		
α	50	0.823	0.440	-0.176	0.421	0.239	-0.078
	100	0.898	0.335	-0.101	0.462	0.190	-0.037
	200	0.921	0.217	-0.078	0.463	0.131	-0.036
	400	0.932	0.153	-0.067	0.478	0.092	-0.021
	800	0.955	0.107	-0.044	0.485	0.064	-0.014
β	50	2.034	1.296	0.534	1.959	0.434	0.159
	100	1.735	0.685	0.235	1.900	0.272	0.100
	200	1.610	0.272	0.110	1.850	0.167	0.050
	400	1.574	0.166	0.074	1.829	0.114	0.029
	800	1.545	0.106	0.045	1.832	0.104	0.032
δ	50	1.348	0.542	-0.151	1.777	0.415	-0.022
	100	1.424	0.434	-0.075	1.756	0.271	-0.043
	200	1.421	0.278	-0.078	1.771	0.192	-0.028
	400	1.424	0.201	-0.075	1.774	0.134	-0.025
	800	1.434	0.152	-0.065	1.772	0.126	-0.027
c	50	0.569	0.547	0.269	0.392	0.247	0.092
	100	0.451	0.372	0.151	0.356	0.130	0.056
	200	0.384	0.163	0.084	0.336	0.086	0.036
	400	0.367	0.110	0.067	0.323	0.055	0.023
	800	0.349	0.074	0.049	0.324	0.053	0.024

$$\times\left[1-\exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right)\right]^{\theta-1},$$

$$f_{BOL-U}(x;a,b,\lambda,\theta) = \frac{1}{B(a,b)}\left[1-\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{a-1}$$

$$\times\left[\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{b-1}$$

$$\times \frac{\lambda^2}{(1+\lambda)} \frac{\theta^2}{(\theta - x)^3} \exp\left\{-\lambda \frac{x}{(\theta - x)}\right\}.$$

For the ELOLLOW distribution, we considered the case when $\alpha = 1$.

6.1 Cancer patients data

We fit the EHL-LLoGW distribution to the data set on remission times (months) of 128 bladder cancer patients (Lee et al. (2003) for details). The observations are as follows: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26 ,26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

Table 4: Parameter estimates for various models fitted for cancer patients data set

	Estimates				
Model	α	β	δ	c	$-2\log L$
EHL-LLoGW	0.8611	0.5191	6.7157	0.0316	819.4
	(0.5676)	(0.1368)	(4.2266)	(0.3098)	
	a	b	α	β	
BW	2.7348	0.9082	0.3216	0.6661	821.7
	(1.6011)	(1.5188)	(0.4382)	(0.2455)	
	a	b	α	β	
KwW	25.6701	26.2789	24.3275	0.1398	821.5
	(4.1821)	(0.1662)	(22.6131)	(0.1417)	
	β	λ	θ	γ	
ELOLLW	6.1173×10^{-5}	0.0378	2.9053	1.0478	828.2
	(0.4015)	(0.0061)	(8.4383×10^{-5})	(0.0676)	
	a	b	α	θ	
TL-WLx	0.5080	0.2070	1.0718	1.5290	820.8
	(0.4034)	(0.2299)	(0.6874)	(1.1199)	
	a	b	λ	θ	
BOL-U	1.1799	2.8446	3.2800×10^{5}	7.6115×10^{6}	826.6
	(0.1329)	(0.3599)	(3.1630×10^{-6})	(1.3631×10^{-7})	

Table 5: Goodness-of-fit statistics for various models fitted for cancer patients data set

Model	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
EHL-LLoGW	827.8	828.1	839.2	0.0235	0.1530	0.0352	0.9974
BW	829.7	830.0	841.1	0.0436	0.2882	0.04495	0.9582
KwW	829.3	829.6	840.7	0.0452	0.2994	0.0471	0.9392
ELOLLW	836.2	836.5	847.6	0.1314	0.7864	0.0670	0.5573
TL-WLx	828.8	829.1	840.2	0.0372	0.2469	0.04287	0.9727
BOL-U	834.6	835.0	846.0	0.1186	0.7118	0.0734	0.4955

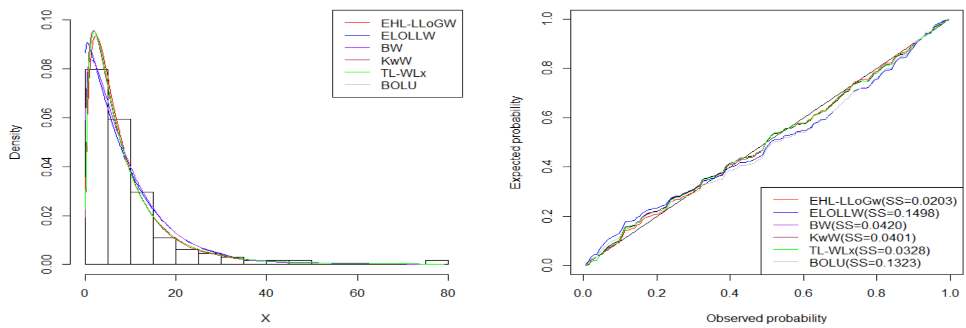

Figure 2: Plots of the fitted curves and probabilities for cancer patients data.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.3222 & -0.0762 & 2.3705 & -0.1605 \\ -0.0762 & 0.0187 & -0.5568 & 0.0357 \\ 2.3705 & -0.5568 & 17.8644 & -1.1712 \\ -0.1605 & 0.0357 & -1.1712 & 0.0960 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$$\alpha \in [0.8611 \pm 1.1125], \beta \in [0.5191 \pm 0.2681], \delta \in [6.7157 \pm 8.2842], c \in [0.0316 \pm 0.6072].$$

The results shown in Table 5 confirm that the EHL-LLoGW model is a better model compared to the other models considered in this paper since it has the lowest values for the goodness-of-fit statistics and the highest p-value for the K-S statistic. Thus, we conclude that the EHL-LLoGW model fit the cancer patients data better than the other models BW, KwW, ELOLLW, TL-WLx and BOL-U.

6.2 Silicon nitride data

The second data set represents fracture toughness of silicon nitride measured in MPa $m^{1/2}$. The data set was also analyzed by Chipepa et al. (2020). The data are: 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

The estimated variance-covariance matrix for model based on silicon nitride data set is

$$\begin{bmatrix} 7.3626 \times 10^{-9} & 7.1721 \times 10^{-7} & 5.8061 \times 10^{-7} & -4.1182 \times 10^{-6} \\ 7.1721 \times 10^{-7} & 3.3731 \times 10^{-4} & 2.7307 \times 10^{-4} & -1.9368 \times 10^{-3} \\ 5.8062 \times 10^{-7} & 2.7307 \times 10^{-4} & 2.2107 \times 10^{-4} & -1.5679 \times 10^{-3} \\ -4.1182 \times 10^{-6} & -1.9368 \times 10^{-3} & -1.5679 \times 10^{-3} & 1.1121 \times 10^{-2} \end{bmatrix}$$

Table 6: Parameter estimates for various models fitted for silicon nitride data set

	Estimates				
Model	α	β	δ	c	$-2\log L$
EHL-LLoGW	8.1×10^{-4}	4.6814	6.9829	1.3671	335.0
	(8.6×10^{-05})	(1.8×10^{-02})	($1.5\times10-02$)	(1.1×10^{-01})	
	a	b	α	β	
BW	0.8013	12.3333	0.01298	5.6883	337.1
	(0.3130)	($5.2\ \times10^{-4}$)	(0.0056)	(1.4291)	
	a	b	α	β	
KwW	0.95740	447.3400	0.0620	5.1840	337.4
	(0.8392)	($8.7\ \times10^{-5}$)	($5.3\ \times10^{-3}$)	(1.7847)	
	β	λ	θ	γ	
ELOLLW	1.0558	0.2683	0.6356	4.2260	336.4
	(0.3108)	(0.0703)	(0.5092)	(0.5822)	
	a	b	α	θ	
TLWLx	1.4694	0.1100	4.8479	0.8960	337.1
	(1.8065)	(0.1677)	(1.9954)	(0.3688)	
	a	b	λ	θ	
BOLU	11.2231	108.6889	0.7531	19.9588	347.3
	(1.2051)	(0.0046)	(0.1081)	(1.2296)	

Table 7: Goodness-of-fit statistics for various models fitted for silicon nitride data set

Model	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
EHL-LLoGW	343.0	343.4	354.1	0.0602	0.3585	0.0568	0.8368
BW	345.1	345.4	356.2	0.0830	0.5025	0.0689	0.6141
KwW	345.4	345.8	356.6	0.0927	0.5689	0.0716	0.5647
ELOLLW	344.4	344.8	355.6	0.0698	0.4309	0.0628	0.7363
TLWLx	345.1	345.5	356.3	0.0828	0.5025	0.0692	0.6182
BOLU	355.3	355.6	366.4	0.2288	1.4031	0.0844	0.364

and the 95% confidence intervals for the model parameters are

$$\alpha \in [8.0851 \times 10^{-4} \pm 0.0002], \quad \beta \in [4.6814 \pm 0.0360],$$
$$\delta \in [6.9829 \pm 0.0291], \quad c \in [1.3671 \pm 0.2067].$$

Furthermore, based on results shown in Table 7, the EHL-LLoGW has the smallest values for the goodness-of-fit statistics and the highest P-value for the K-S statistic compared to the other models considered. Therefore, we conclude that the EHL-LLoGW model fit the silicon nitride data better than the non-nested models BW, KwW, ELOLLW, TL-WLx and BOL-U distributions. Also, Figure 3 show that our proposed model fit the silicon nitride data set better than the selected competing models.

6.3 Kevlar 49/epoxy strands failure at 90% data

The third data set is reported by Andrews et al. (2012) and also by Barlow et al. (1984), which represents failure times (in hours) of kevlar 49/epoxy strands subjected to constant sustained pressure at the 90% stress level. The data set was also analyzed by Chipepa et al. (2020). The observations are as follows: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13,

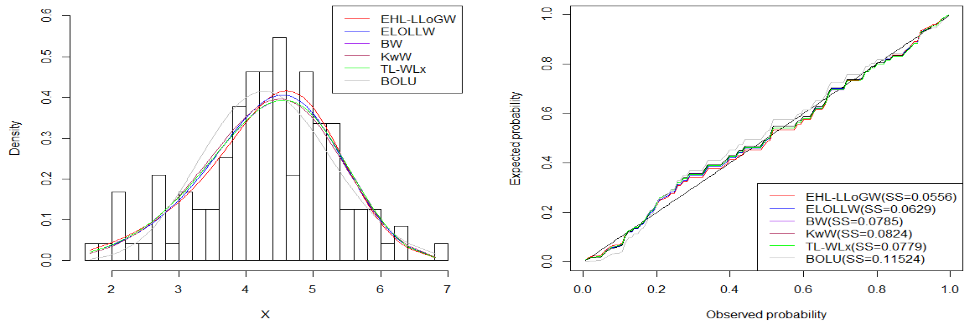

Figure 3: Plots of the fitted curves and probabilities for silicon nitride data.

0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89. The estimated variance-covariance matrix for EHL-LLoGW model based on kevlar data set is given by

$$\begin{bmatrix} 0.2340 & -0.1059 & 0.2550 & -0.1957 \\ -0.1059 & 0.0624 & -0.1114 & 0.0769 \\ 0.2550 & -0.1114 & 0.3025 & -0.2344 \\ -0.1957 & 0.0769 & -0.2344 & 0.2010 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are
$\alpha \in [0.8574 \pm 0.9482]$, $\beta \in [0.9914 \pm 0.4895]$, $\delta \in [1.1002 \pm 1.0779]$ and $c \in [0.6497 \pm 0.8788]$.

Table 8: Parameter estimates for various models fitted for kevlar data set

	Estimates				
Model	α	β	δ	c	$-2\log L$
EHL-LLoGW	0.8575	0.9914	1.1002	0.6497	204.5
	(0.4838)	(0.2497)	(0.5500)	(0.4484)	
	a	b	α	β	
BW	0.7485	572.05	0.0023	1.1084	205.5
	(0.3056)	(1.0355 $\times 10^{-8}$)	(0.0026)	(0.2940)	
	a	b	α	β	
KwW	19.4050	2.4710 $\times 10^{4}$	0.2620	0.0762	206.9
	(5.8370)	(8.1472)	(0.2491)	(0.5153)	
	β	λ	θ	γ	
ELOLLW	7.0692	0.1556	7.3936	0.8374	205.2
	(4.1287)	(0.1115)	(3.9446)	(0.1094)	
	a	b	α	θ	
TL-WLx	0.6616	0.7096	1.3633	0.6079	205.4
	(0.6595)	(0.9639)	(0.9469)	(0.4456)	
	a	b	λ	θ	
BOL-U	0.8718	7.3537	3.0051 $\times 10^{4}$	2.5755 $\times 10^{5}$	205.7
	(0.1068)	(1.1462)	(2.7847 $\times 10^{-4}$)	(3.2491 $\times 10^{-5}$)	

Table 9: Parameter estimates and goodness-of-fit statistics for various models fitted for kevlar data set

Model	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
EHL-LLoGW	212.5	212.9	223.0	0.1350	0.8153	0.0743	0.6326
BW	213.5	213.9	224.0	0.1613	0.9408	0.0834	0.4834
KwW	214.9	215.3	225.3	0.2405	1.3119	0.0983	0.2835
ELOLLW	213.2	213.6	223.7	0.1666	0.9601	0.0816	0.5123
TL-WLx	213.4	213.8	223.9	0.1625	0.9447	0.0835	0.4825
BOL-U	213.7	214.1	224.1	0.1819	1.0330	0.0896	0.3927

Results shown in Table 9 further confirms that the EHL-LLoGW model is indeed a better model compared to the other selected models. Thus, we conclude that the EHL-LLoGW model fit the kevlar data better than the other models: BW, KwW, ELOLLW, TL-WLx and BOL-U distributions. Also, Figure 4 show that our proposed model performs better than the competing models on kevlar data.

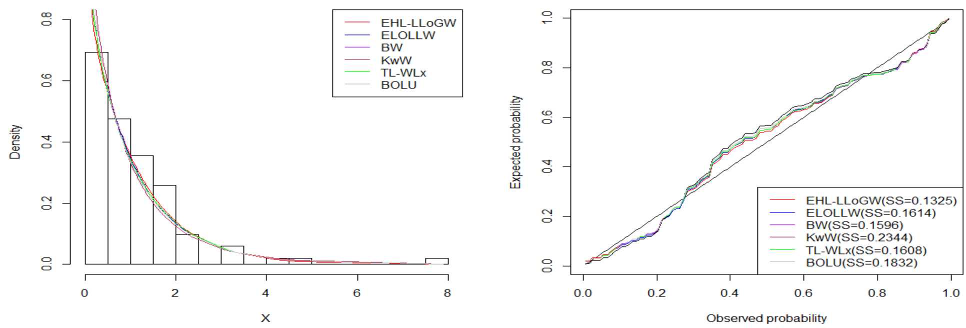

Figure 4: Plots of the fitted curves and probabilities for kevlar data.

7 Conclusions

We have presented a generalized distribution, referred to as the Exponentiated Half Logistic-Log-Logistic Weibull (EHL-LLoGW) distribution. Statistical properties of the EHL-LLoGW distribution were also derived. We obtained maximum likelihood estimates of the parameters of the EHL-LLoGW distribution. A simulation study was conducted to assess the consistency of the maximum likelihood estimates. The applications provided showed that EHL-LLoGW distribution is suitable for modeling heavy tailed and almost symmetric data sets.

References

Afify, A.Z., Altun, E., Alizadeh, M., Ozel, G. and Hamedani G.G. (2017). The odd exponentiated half-logistic-G family: properties, characterization and applications. *Chilean Journal of Statistics*, **8**(2), 65-91.

Aldahlan, M. and Afify, A.Z. (2018). The odd exponentiated half-logistic Burr XII distribution. *Pakistan Journal of Statistics and Operation Research*, **14**(2), 305-317.

Afify, A.Z., Alizadeh, M., Zayed, M., Ramires, T.G. and Louzada, F. (2018). The odd log-logistic exponentiated Weibull distribution: regression modeling, properties, and applications. *Iranian Journal of Science and Technology*, **42**(4), 2273-2288.

Andrews, D.F. and Herzberg, A. M. (2012). *Data: A collection of problems from many fields for the student and research worker*. Springer Science & Business Media.

Barlow, R.E., Toland, R.H., and Freeman, T. (1984). A Bayesian analysis of stress-rupture life of Kevlar/Epoxy spherical pressure vessels. *Proceedings of the Canadian Conference in Applied Statistics*, Edited by: Dwivedi, T. D. New York: Marcel Dekker.

Balakrishnan, N. (1985). Order statistics from the half logistic distribution. *Journal of Statistical Computation and Simulation*, **20**, 287-309.

Chambers, J., Cleveland, W., Kleiner, B. and Tukey, P. (1983). *Graphical methods of data analysis*. Chapman and Hall.

Chen, G. and Balakrishnan, N. (1985). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, **27**(2), 154-161.

Chipepa, F., Oluyede, B. and Makubate, B. (2020). The odd generalized half logistic Weibull-G family of distributions with applications. Journal of Statistical Modeling Theory and Applications, (In Press).

Cordeiro, G.M., Alizadeh, M. and Diniz Marinho, P.R. (2016). The type I half-logistic family of distributions. *Journal of Statistical Computation and Simulation*, **86**(4): 707-728.

Cordeiro, G.M., Alizadeh, M. and Ortega, E.M.M. (2014). The exponentiated half-logistic family of distributions: properties and applications. *Journal of Probability and Statistics*, **2014**, Article ID 864396, 21 pages. doi:10.1155/2014/86439

Cordeiro, G.M., Alizadeh, M. and Diniz Marinho, P.R. (2016). The type I half-logistic family of distributions. *Journal of Statistical Computation and Simulation*, **86**(4): 707-728.

Cordeiro, G.M., Gomes, A., da Silva, C. and Ortega, E.M.M. (2013). The beta exponentiated Weibull distribution. *Journal of Statistical Computation and Simulations*, **38**(1), 114-138.

Cordeiro, G.M. Ortega, E.M.M. and Nadarajaah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, **347**, 1399 -1429.

Gupta, R.D., and Kundu, D. (2001). Exponentiated Exponential Distribution: An alternative to gamma and Weibull distributions. *Biometrical Journal*, **43**, 117-130.

Jamal, F., Reyad, H. M., Nasir, M.A., Chesneau, C., Shah, M.A.A. and Ahmed S.O. (2019). Topp-Leone Weibull-Lomax distribution: properties, regression model and applications, hal-02270561.

Korkmaz, M.C., Yousof, H.M. and Hamedani, G.G. (2018). The exponential Lindley odd log-logistic-G family: properties, characterizations and applications. *Journal of Statistical Theory and Applications*, **17**(3), 554-571.

Lai, C.D., Xie, M., and Murthy, D.N.P. (2003). A modified Weibull distribution. *IEEE Transactions on Reliability*, **52**(1), 33-37.

Lee, E.T. and Wang, J.W. (2003). *Statistical methods for survival data analysis*. 3rd edition, Wiley, Hoboken.

Nadarajah, S., Cordeiro, G.M., and Ortega, E.M.M. (2011). General results for the beta-modified Weibull distribution. *Journal of Statistical Computation and Simulations*, **81**(10), 1211-1232.

Oluyede, B., Foya, S.,Warahena-Liyanage, G., and Huang, S., (2016). The log-logistic Weibull distribution with applications to lifetime data. *Austrian Journal of Statistics*, **45**(2), 43-69.

Oluyede, B.O., Pu, S., Makubate, S., and Qui, Y. (2018). The gamma Weibull-G family of distributions. *Austrian Journal of Statistics*, **47**, 45-76.

Oluyede, B.O., Huang, S., and Yang, T. (2015). A new class of generalized modified Weibull distributions with applications. *Austrian Journal of Statistics*, **44**, 45-68.

Rényi, A. (1960). On measures of entropy and information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 547 - 561.

Shannon, C.E. (1951). Prediction and entropy of printed english. *The Bell System Technical Journal*, **30**, 50-64.

Yousof, H.M., Alizadeh, M., Jahanshahi, S.M.A., Ramires, T.G., Ghosh, I. and Hamedani, G.G. (2017). The transmuted Topp-Leone-G family of distributions: theory, characterizations and applications. *Journal of Data Science*, **15**, 723-740.

Appendix

The following URL contains all computational codes.
https://drive.google.com/drive/folders/1eU42GCNBMCMDhpkcXrhVxZsZek-8c3Nj?usp=sharing