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Research Paper

The hyperbolic Tan-X family of distributions: Properties, application and characterization

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Abstract: In this paper, we introduce the hyperbolic tan-X family of distributions as a new statistical distribution defined by a trigonometric function. Some properties and applications of the new family are investigated in some detail. A characterization theorem extending the new class of statistical distributions is also considered. Avenues for further research are also presented.

Keywords: Breaking stress of carbon fibers data; Hyperbolic functions; Statistical distributions; Tangent; Trignometric functions.
Mathematics Subject Classification (2010): 62Exx

1 Introduction

Statistical distributions defined by trigonometric functions have received considerable attention in the literature. In Mahmood et al. (2019) they introduced the N-sine-G family of distributions based on a sine transformation. In Kumar et al. (2015), a new lifetime distribution is proposed by the use of the Sine function in terms of some lifetime distribution as baseline distribution. In Jamal and Chesneau (2019), the new family is defined by a quotient of two functions using polynomial, exponential, sine, and cosine functions. Further studies include the cosine distribution (Raab and Green, 1961), the beta trigonometric distribution (Nadarajah and Kotz, 2006), the Von-Mises distribution (Evans et al., 2000; Strukov, 2013), the sine square distribution (Al-Farris and Khan, 2008), and the sin-skew logistic distribution (Chakraborty et al., 2012).

In this study, we continue to add to the class of statistical distributions, that can be defined using trigonometric functions, we propose the class of hyperbolic tan-X

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distributions, and study in some detail their properties, with applications. A characterization extending the new family is also considered. As the class of statistical distributions that can be defined by trigonometric functions is limited, the main aim in introducing this new family is to provide the researcher with more choice when it comes to selecting a particular trigonometric distribution. The hope is that this class of distributions will provide consistently better fits, than other distributions in the statistical modeling of data.

This paper is organized as follows. In Section 2, the probability density function (PDF) and the cumulative density function (CDF) of the new family are defined. In Section 3, a sub-model of the new family is presented, namely hyperbolic tan-normal, and the PDF, CDF, hazard function (HF), and survival function (SF) are visualized. Section 4 is devoted to some statistical properties of the new family, there we obtain an expansion for the PDF and CDF, the quantile function, rth non-central moments, Renyi entropy, and moment generating function. In Section 5, the method of maximum likelihood in estimating model parameters is discussed, and in Section 6 a Monte Carlo simulation study is carried out to assess the performance of the method of maximum likelihood. In section 7, we demonstrate the usefulness of the new family to a data set coming from the material and chemical engineering sciences. A characterization theorem extending the new family in terms of the hazard rate function of a random variable is presented in Section 8. The paper is concluded in Section 9, where we recommend to the reader to obtain some properties and applications of the exponentiated hyperbolic tan-X family of distributions.

2 A new family

A random variable J will be said to follow the hyperbolic tan-X family of distributions if its CDF is given by

$$G(x;\boldsymbol{\xi}) = \mathrm{Tanh}(3\pi F(x;\boldsymbol{\xi})),$$

where $x \in Supp(F)$, and $\boldsymbol{\xi}$ is a vector of parameters in the baseline distribution depending on F. The PDF of the new family is given by

$$g(x;\boldsymbol{\xi}) = 3\pi f(x;\boldsymbol{\xi})\mathrm{sech}^2(3\pi F(x;\boldsymbol{\xi})),$$

where f is the PDF of the baseline distribution.

To define the new distribution, we modified the argument of Tanh(x), so that the resulting function is a proper distribution function.

3 A sub-model of the new family

The sub-model is called the hyperbolic tan-normal family of distributions. The PDF is given by

$$f(x;a,b) = \frac{3\sqrt{\frac{\pi}{2}}e^{-\frac{(a-x)^2}{2b^2}}\mathrm{sech}^2\left(\frac{3}{2}\pi\mathrm{erfc}\left(\frac{a-x}{\sqrt{2}b}\right)\right)}{b},$$

the CDF is given by

$$F(x;a,b) = \tanh\left(\frac{3}{2}\pi\text{erfc}\left(\frac{a-x}{\sqrt{2}b}\right)\right).$$

The survival function (SF) is given by

$$S(x;a,b) = 1 - \tanh\left(\frac{3}{2}\pi\text{erfc}\left(\frac{a-x}{\sqrt{2}b}\right)\right),$$

and the hazard rate function (HF) is given by

$$H(x;a,b) = \frac{3\sqrt{\frac{\pi}{2}}e^{-\frac{(a-x)^2}{2b^2}}\text{sech}^2\left(\frac{3}{2}\pi\text{erfc}\left(\frac{a-x}{\sqrt{2}b}\right)\right)}{b\left(1-\tanh\left(\frac{3}{2}\pi\text{erfc}\left(\frac{a-x}{\sqrt{2}b}\right)\right)\right)},$$

where $\text{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ is the complementary error function. We write $K \sim HTN(a,b)$ if the random variable K follows the hyperbolic tan-normal family of distributions.

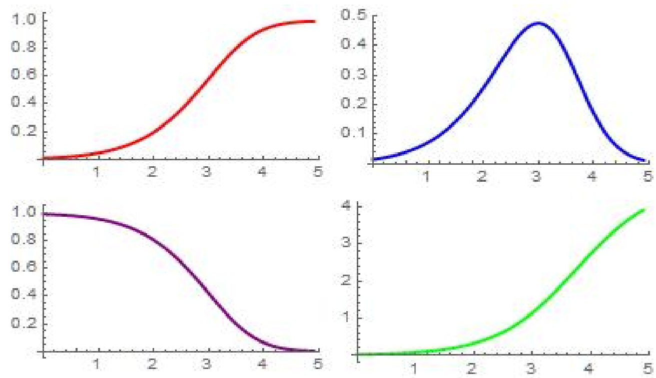

Figure 1: The CDF (red), PDF(blue), SF(purple), and HF(green) of HTN(5.60249,1.76943).

4 Some mathematical properties

4.1 Expansion for the CDF

Since $\text{Tanh}(x) = \frac{e^{2x}-1}{e^{2x}+1}$, then using the power series representation of the exponential function we can write $e^{2x} = \sum_{q=0}^{\infty}\frac{(2x)^q}{q!}$. Therefore, $\text{Tanh}(x) = \frac{\sum_{q=0}^{\infty}\frac{(2x)^q}{q!}-1}{\sum_{q=0}^{\infty}\frac{(2x)^q}{q!}+1}$. Hence,

$$G(x;\boldsymbol{\xi}) = \text{Tanh}(3\pi F(x;\boldsymbol{\xi})) = \frac{\sum_{q=0}^{\infty}\frac{(6\pi F(x;\boldsymbol{\xi}))^q}{q!}-1}{\sum_{q=0}^{\infty}\frac{(6\pi F(x;\boldsymbol{\xi}))^q}{q!}+1}.$$

4.2 Expansion for the PDF

Since $\text{Sech}(x) = \frac{2e^x}{e^{2x}+1}$, then,

$$g(x;\boldsymbol{\xi}) = 3\pi f(x;\boldsymbol{\xi})\text{sech}^2(3\pi F(x;\boldsymbol{\xi})) = \frac{12\pi f(x;\boldsymbol{\xi})e^{6\pi F(x;\boldsymbol{\xi})}}{(e^{6\pi F(x;\boldsymbol{\xi})}+1)^2}.$$

By using the negative binomial series and the power series representation for the exponential function, we can write

$$g(x;\boldsymbol{\xi}) = \sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{k,q}f(x)F(x)^q,$$

where $\Omega_{k,q} = 12\pi(-1)^k\binom{1+k}{k}\frac{(6\pi(k+1))^q}{q!}$.

4.3 Quantile function

Let $0 < p < 1$. we must solve the equation $p = \text{Tanh}(3\pi F(Q(p)))$ with respet to $Q(p)$. This yields

$$Q(p) = F^{-1}\left[\frac{1}{3\pi}\text{Tanh}^{-1}(p)\right],$$

where F^{-1} is the quantile function of the baseline distribution with CDF $F(x)$.

4.4 Moments

Observe that if U is uniform on $(0,1)$, then the random variable

$$X = F^{-1}\left[\frac{1}{3\pi}\text{Tanh}^{-1}(U)\right],$$

follows the hyperbolic tan-x family of distributions. According to Nasiru et al. (2017), we can write

$$Q_X(u) = \sum_{i=0}^{\infty}h_iu^i,$$

where the coefficients are suitably chosen real numbers that depend on the parameters of the $F(x)$ distribution. For a power series raised to a positive integer $r \geq 1$, we have

$$(Q_X(u))^r = \left(\sum_{i=0}^{\infty}h_iu_i\right)^r = \sum_{i=0}^{\infty}\delta_{r,i}u^i,$$

where $\delta_{r,i}$ are obtained from $\delta_{r,i} = (ih_0)^{-1}\sum_{s=1}^{i}[s(r+1)-i]h_s\delta_{r,i-s}$ with $\delta_{r,0} = h_0^r$ for $i = 1,2,\cdots$ (Gradshteyn et al, 2007). Thus, we have

$$\mu_r' = \sum_{i=0}^{\infty}\delta_{r,i}E\left[\left(\frac{1}{3\pi}\text{Tanh}^{-1}(U)\right)^i\right],$$

where $E(\cdot)$ is an expectation. Since $\mathrm{Tanh}^{-1}(U) = \frac{1}{2}\ln\left(\frac{1+U}{1-U}\right)$, we can observe

$$\left(\frac{1}{3\pi}\mathrm{Tanh}^{-1}(U)\right)^i = \left(\frac{1}{6\pi}\right)^i\left(\ln\left(1-\frac{2U}{U-1}\right)\right)^i.$$

By equation (3.18), (Almheidat et al., 2015) we can write

$$\left(\ln\left(1-\frac{2U}{U-1}\right)\right)^i = (-1)^{-i} i\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{(-1)^{m+j}}{i-j}\binom{m-i}{m}\binom{m}{j}\rho_{j,m}\left(\frac{2U}{U-1}\right)^{i+m},$$

where $\rho_{j,m}$ are calculated from equation (3.19) (Almheidat et al., 2015). Put

$$\Omega_{i,m,j} = \delta_{r,i}\left(\frac{1}{6\pi}\right)^i(-1)^{-i} i\frac{(-1)^{m+j}}{i-j}\binom{m-i}{m}\binom{m}{j}\rho_{j,m}.$$

Then,

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\Omega_{i,m,j}E\left[\left(\frac{2U}{U-1}\right)^{i+m}\right].$$

4.5 Renyi entropy

Since

$$g(x;\boldsymbol{\xi}) = 3\pi f(x;\boldsymbol{\xi})\mathrm{sech}^2(3\pi F(x;\boldsymbol{\xi})) = \frac{12\pi f(x;\boldsymbol{\xi})e^{6\pi F(x;\boldsymbol{\xi})}}{(e^{6\pi F(x;\boldsymbol{\xi})}+1)^2},$$

we first find an expansion for $g(x)^\delta$ where $\delta > 0$, and $\delta \neq 1$. By the negative binomial series we can write

$$(e^{6\pi F(x)}+1)^{-2\delta} = \sum_{q=0}^{\infty}(-1)^q\binom{2\delta+q-1}{q}e^{6\pi F(x)q}.$$

By the power series representation for the exponential function, we can write $e^{6\pi F(x)q} = \sum_{m=0}^{\infty}\frac{(6\pi F(x)q)^m}{m!}$. We can also write $e^{6\pi F(x)\delta} = \sum_{k=0}^{\infty}\frac{(6\pi F(x)\delta)^k}{k!}$. Put

$$\Omega_{k,q,m} = (12\pi)^\delta\frac{(6\pi\delta)^k}{k!}(-1)^q\binom{2\delta+q-1}{q}\frac{(6\pi q)^m}{m!},$$

then

$$g(x)^\delta = \sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\Omega_{k,q,m}f(x)^\delta F(x)^{k+m}.$$

Thus, the Renyi entropy is

$$I_R(\delta) = \frac{1}{1-\delta}\log\left(\sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\Omega_{k,q,m}\int_{-\infty}^{\infty}f(x)^\delta F(x)^{k+m}dx\right).$$

4.6 Moment generating function

Given a random variable X, the moment generating function is defined as

$$M_X(z) = E[e^{zX}]$$

where $E[\cdot]$ is an expectation. Using the series expansion for e^{zX}, one can write $M_X(z) = \sum_{r=0}^{\infty} \frac{z^r \mu'_r}{r!}$, where μ'_r is the r^{th} non-central moment of the random variable X. Thus, the moment generating function of the hyperbolic tan-X family of distributions is given by

$$\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{m} \frac{z^r \Omega_{i,m,j}}{r!} E\left[\left(\frac{2U}{U-1}\right)^{i+m}\right],$$

where U is uniform random variable on $(0,1)$ and $\Omega_{i,m,j}$ is defined as in Section 5.4

5 Parameter estimation

The method of maximum likelihood is used in this paper to estimate model parameters. Here we discuss this method for the hyperbolic tan-X family of distributions. Suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from the hyperbolic tan-X family of distributions. It can be shown that the total log-likelihood function is given by

$$\ln L = \sum_{i=1}^{n} \left\{ \ln(3\pi) + \ln f(x_i; \boldsymbol{\xi}) + 2\ln(\sec(3\pi F(x_i; \boldsymbol{\xi}))) \right\},$$

where $\boldsymbol{\xi}$ is a vector of parameters associated with the baseline distribution. Partial differentiation of the total log-likelihood function with respect to model parameters gives the following as the score function

$$\frac{\partial \ln L}{\partial \boldsymbol{\xi}} = \sum_{i=1}^{n} \left\{ \frac{\frac{\partial f(x_i, \boldsymbol{\xi})}{\partial \boldsymbol{\xi}}}{f(x; \boldsymbol{\xi})} + 6\pi f(x_i; \boldsymbol{\xi}) \tan(3\pi F(x_i; \boldsymbol{\xi})) \right\}.$$

Equating the score function to zero and numerically solving the equation using techniques such as the quasi Newton-Raphson method, gives the maximum likelihood estimates for the model parameters. Let $\triangle = (\boldsymbol{\xi})$, to construct confidence intervals for the parameters in the hyperbolic tan-X family of distributions, the observed information matrix, call it $J(\triangle)$, can be used due to the complex nature of the expected information matrix. The observed information matrix is given by

$$J(\triangle) = -\left[\frac{\partial^2 lnL}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}}\right].$$

When the usual regularity conditions are satisfied and that the parameters are within the interior of the parameter space, but not on the boundary, the distribution of $\sqrt{n}(\widehat{\triangle} - \triangle)$ converges to the multivariate normal distribution $N_p(0, I^{-1}(\triangle))$, where $I(\triangle)$ is the expected information matrix, and it is assumed that $\boldsymbol{\xi} = (\boldsymbol{\xi_1}, \cdots, \boldsymbol{\xi_p})$. The asymptotic behavior remains valid when $I(\triangle)$ is replaced by the observed information matrix evaluated at $J(\widehat{\triangle})$. The asymptotic multivariate normal distribution $N_p(0, J^{-1}(\widehat{\triangle}))$ is a very useful tool for constructing an approximate $100(1-\psi)\%$ two-sided confidence intervals for the model parameters, where ψ is the significance level.

6 Monte Carlo simulation study

In this section, we show that the method of maximum likelihood is adequate in estimating the parameters in the hyperbolic tan-X family. For this, a Monte Carlo simulation study is carried out to assess the performance of the estimation method in the hyperbolic tan-normal (HTN) sub-model. Samples of sizes 200, 400, 500, and 700, are drawn from the HTN distribution, and the samples have been drawn for the following set of parameters

(*a*) Set I: $(a, b) = (0.3, 0.5)$

(*b*) Set II: $(a, b) = (0.5, 0.3)$

The maximum likelihood estimators for the parameters alpha and beta are obtained. The procedure has been repeated 400 times, and the mean and standard deviation for the estimates are computed, and the results are summarized in Tables 1 and 2 for each of sets I and II, respectively. From these tables, we can find that the simulated estimates are close to the true values of the parameters and hence the estimation method is adequate. We have also observed that the estimated standard deviation consistently decreases with increasing sample size as can been seen by plotting the standard deviation against the sample size. Overall the simulation study conducted, indicated that using the method of maximum likelihood in estimating model parameters is adequate.

Table 1: Result of simulation study for set I.

Parameter	Sample Size	Average Estimate	Standard Deviation
a	200	0.294	0.044
	400	0.298	0.028
	500	0.298	0.026
	700	0.298	0.020
b	200	0.497	0.028
	400	0.499	0.018
	500	0.499	0.017
	700	0.499	0.014

Table 2: Result of simulation study for set II.

Parameter	Sample Size	Average Estimate	Standard Deviation
a	200	0.496	0.026
	400	0.499	0.016
	500	0.499	0.015
	700	0.498	0.012
b	200	0.298	0.017
	400	0.299	0.011
	500	0.299	0.010
	700	0.299	0.008

7 Applications

Here we demonstrate the usefulness of the new family to the breaking stress of carbon fibers data, Table 2 (Alzaatreh et al., 2014). The PDF and CDF of the submodel of the new family that we consider is already given in Section 3. The other competing models are the Hyperbolic Sine-Weibull (HSW), and the hyperbolic cosine-logistic distribution (HCL). For the HSW distribution, the PDF is

$$w(x;a,b,c) = \frac{2abcx^{b-1}e^{a-cx^b}\sinh\left(a\left(1-e^{-cx^b}\right)\right)}{(e^a-1)^2},$$

the CDF is

$$W(x;a;b,c) = \frac{2e^a\left(\cosh\left(a\left(1-e^{-cx^b}\right)\right)-1\right)}{(e^a-1)^2},$$

the survival function (SF) is given by

$$SW(x;a,b,c) = 1-\frac{2e^a\left(\cosh\left(a\left(1-e^{-cx^b}\right)\right)-1\right)}{(e^a-1)^2},$$

and hazard function (HF) is given by

$$Hw(x;a,b,c) = \frac{2abcx^{b-1}e^{a-cx^b}\sinh\left(a\left(1-e^{-cx^b}\right)\right)}{(e^a-1)^2\left(1-\frac{2e^a\left(\cosh\left(a\left(1-e^{-cx^b}\right)\right)-1\right)}{(e^a-1)^2}\right)}.$$

We write $P \sim HSW(a,b,c)$, if P is a hyperbolic sine-Weibull random variable.

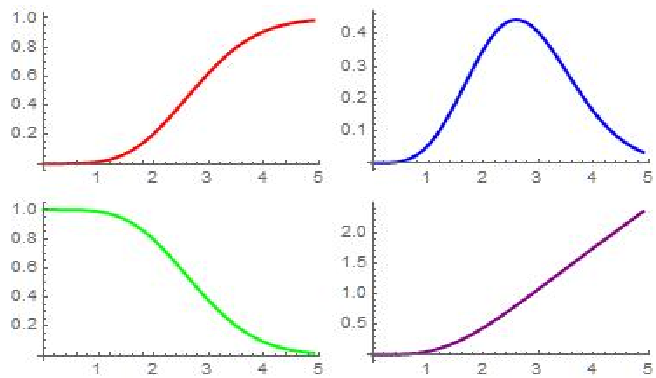

Figure 2: The CDF (red), PDF(blue), SF(green), and HF(purple) of HSW(0.0000304063, 2.34175, 0.119485)

For the hyperbolic cosine-logistic distribution, the CDF is given by

$$L(x;a,c,d) = \frac{2e^a\sinh\left(\frac{a}{e^{-\frac{x-c}{d}}+1}\right)}{e^{2a}-1},$$

the PDF given by

$$l(x;a,c,d)=\frac{2ae^{a-\frac{x-c}{d}}\cosh\left(\frac{a}{e^{-\frac{x-c}{d}}+1}\right)}{\left(e^{2a}-1\right)d\left(e^{-\frac{x-c}{d}}+1\right)^2},$$

the Surival function is given by

$$SL(x;a,c,d)=1-\frac{2e^{a}\sinh\left(\frac{a}{e^{-\frac{x-c}{d}}+1}\right)}{e^{2a}-1},$$

and Hazard function is given by

$$Hl(x;a,c,d)=\frac{2ae^{a-\frac{x-c}{d}}\cosh\left(\frac{a}{e^{-\frac{x-c}{d}}+1}\right)}{\left(e^{2a}-1\right)d\left(e^{-\frac{x-c}{d}}+1\right)^2\left(1-\frac{2e^{a}\sinh\left(\frac{a}{e^{-\frac{x-c}{d}}+1}\right)}{e^{2a}-1}\right)}.$$

We write $K\sim HCL(a,c,d)$, if K is a hyperbolic cosine-logistic random variable.

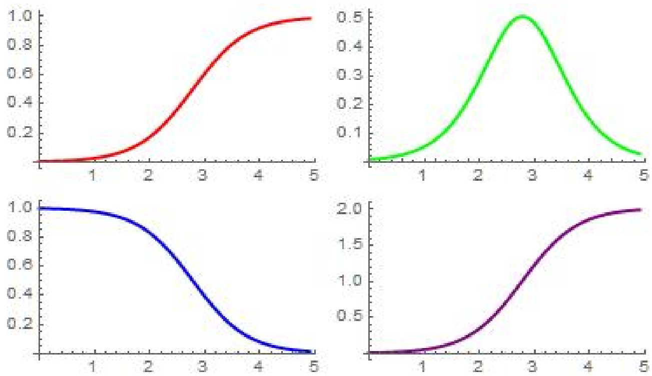

Figure 3: The CDF (red), PDF(green), SF(blue), and HF(purple) of HCL(0.0170707, 2.77939, 0.494615)

Table 3: Estimates for the parameters of fitted distribution

Distribution	Parameters	Estimates	Standard error
HTN	$\hat{a}$	5.603059	0.2602749
	$\hat{b}$	1.769822	0.1690595
HSW	$\hat{a}$	4.2595809	1.8205297
	$\hat{b}$	1.9415779	0.3295281
	$\hat{c}$	0.2628609	0.1352378
HCL	$\hat{a}$	1.9609166	1.18744335
	$\hat{c}$	2.4365866	0.32169502
	$\hat{d}$	0.4851145	0.05256767

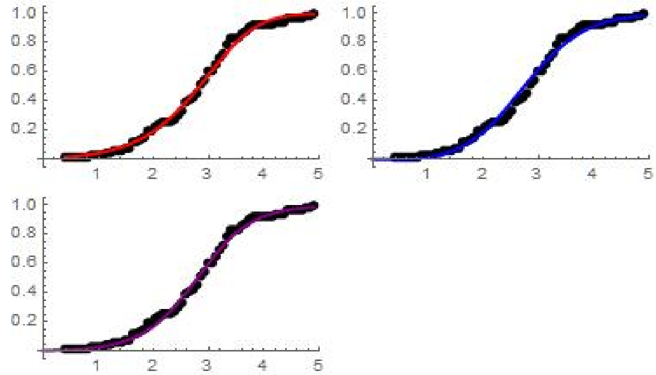

Figure 4: The CDF's of HTN (red), HSW (blue), and HCL (purple) fitted to the empirical distribution of Table 2 (Alzaatreh et.al (2014))

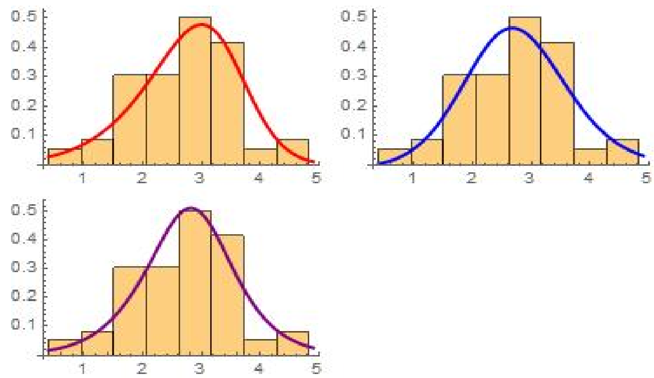

Figure 5: The PDF's of HTN (red), HSW(blue), and HCL (purple) fitted to the histogram of Table 2 (Alzaatreh et.al (2014))

Using the R software, we report the estimates for the parameters in each of the three distributions alongside their standard errors in Table 3. The fitted CDFs to the carbon fibers data are shown in Figures 3 and 4. The measures of goodness of fit we consider include Bayesian information criterion (BIC), negative Log-Likelihood, Cramer von-Misses (W), Anderson Darling (A), KS (Kolmogorov Smirnov), AIC (Akaike's Information Criterion), CAIC (Consistent Akaike's Information Criterion), and HQIC (Hannan-Quinn information criterion), and they are reported in Table 4 below. Whilst it appears from the fits above, that all the distributions are competitive in fitting the breaking stress of carbon fibers data, Table 4 reveals that HCL is slightly better than HTN, but HCL is better than HSW. However, HTN is better than HSW. Over the new family should be highly selective among researchers with an interest in using trigonometric distributions in statistical modeling.

Table 4: Goodness of fit measures.

	HTN	HSW	HCL
W	0.051	0.111	0.046
A	0.398	0.593	0.280
KS statistic	0.073	0.078	0.057
KS p-value	0.864	0.808	0.981
AIC	176.074	179.061	176.404
CAIC	176.265	179.448	176.791
BIC	180.453	185.630	182.973
HQIC	177.805	181.657	179.000
-Log(likelihood)	86.0372	86.530	85.202

8 A characterization theorem

It is well known that the hazard function, h_F, of a twice differentiable function, F, satisfies the first-order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

In this section, we present a Weibull-hyperbolic tan X distribution. The result here is inspired by Alizadeh et al. (2018). First, let us introduce the following.

Definition 8.1. *We say a random variable X follows a Weibull-G distribution if its CDF is given by*

$$F(x;\boldsymbol{\xi}) = 1 - e^{-\left(\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right)^{\alpha}},$$

where G is some baseline distribution, $x \in Supp(G)$, and $\boldsymbol{\xi}$ is a vector of parameters in the baseline distribution whose support depends on G, and $\alpha > 0$, and $\overline{G} = 1 - G$.

The PDF of the Weibull-G distribution is given by

$$f(x;\boldsymbol{\xi}) = \alpha g(x;\boldsymbol{\xi}) \frac{G(x;\boldsymbol{\xi})^{\alpha-1}}{\overline{G}(x;\boldsymbol{\xi})^{\alpha+1}} e^{-\left(\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right)^{\alpha}},$$

where g is the PDF of the baseline distribution. Clearly, the hazard rate function of the Weibull-G distribution is given by

$$h_F(x;\boldsymbol{\xi}) = \alpha g(x;\boldsymbol{\xi}) \frac{G(x;\boldsymbol{\xi})^{\alpha-1}}{\overline{G}(x;\boldsymbol{\xi})^{\alpha+1}}.$$

Theorem 8.2. *Let $X : \Omega \mapsto \mathbb{R}$ be a continuous random variable. The PDF of X is*

$$\alpha g(x;\boldsymbol{\xi}) \frac{G(x;\boldsymbol{\xi})^{\alpha-1}}{\overline{G}(x;\boldsymbol{\xi})^{\alpha+1}} e^{-\left(\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right)^{\alpha}},$$

for some baseline distribution with PDF g, CDF G, $\alpha > 0$, and $\overline{G} = 1 - G$, if and only if its hazard rate function $h_F(x)$ satisfies the following differential equation

$$h'_F(x) - g'(x)g(x)^{-1}h_F(x) = \alpha g(x)\frac{d}{dx}\frac{G(x)^{\alpha-1}}{\overline{G(x)}^{\alpha+1}},$$

with $x \in \mathbb{R}$, with initial condition $h_F(0) = 0$ for $\alpha > 1$.

Proof. If X has PDF as stated in the theorem, then the differential equation as stated in the theorem holds. Now if the stated differential equation holds, then

$$\frac{d}{dx}\left\{g(x)^{-1}h_F(x)\right\} = \alpha\frac{d}{dx}\frac{G(x)^{\alpha-1}}{\overline{G(x)}^{\alpha+1}},$$

or

$$h_F(x;\boldsymbol{\xi}) = \alpha g(x;\boldsymbol{\xi})\frac{G(x;\boldsymbol{\xi})^{\alpha-1}}{\overline{G(x;\boldsymbol{\xi})}^{\alpha+1}},$$

which is the hazard function of Weibull-G □

Characterization of the Weibull-hyperbolic tan X distribution. is obtained from the above theorem by letting the baseline PDF and CDF be given as in Section 2.

9 Conclusions

In this paper, a new class of statistical distribution is introduced. Mathematical properties such as expansion for CDF and PDF, quantile function, rth non-central moments, Renyi entropy, and moment generating function are studied in detail. The method of maximum likelihood is used to estimate model parameters. A simulation study is conducted to determine the effectiveness of the maximum likelihood estimation method. Application to real data is illustrated to show the usefulness of the new family. A characterization theorem extending the new family is also presented.

As a further recommendation, we suggest obtaining some properties and applications of a so-called exponentiated hyperbolic-tan X family of distributions. The CDF of the new family is given by

$$G(x;\boldsymbol{\xi},a) = \tanh^{a}(3\pi F(x;\boldsymbol{\xi})),$$

and the PDF is given by

$$g(x;\boldsymbol{\xi};a) = 3\pi a f(x;\boldsymbol{\xi})\mathrm{sech}^2(3\pi F(x;\boldsymbol{\xi}))\tanh^{a-1}(3\pi F(x;\boldsymbol{\xi})),$$

where the baseline distribution has CDF $F(x)$ and PDF $f(x)$, $a > 0$; $\boldsymbol{\xi}$ is a vector of parameters in the baseline distribution.

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