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# The exponentiated half logistic odd Weibull-Topp-Leone-G: Model, properties and applications

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**Abstract:** A new family of distributions referred to as the exponentiated half logistic odd Weibull-Topp-Leone-G family of distributions is developed. We derive statistical properties of the new family of distributions. The distribution can be expressed as a linear combination of the exponentiated-G distribution. Five special cases for the new family of distributions are also presented. Estimation and Monte Carlo simulation study was also conducted. Two real data examples for a selected special case are also presented.

**Keywords:** Estimation; Exponentiated half logistic odd Weibull-Topp-Leone-G; Half logistic odd Weibull-Topp-Leone-G; Inference; Topp-Leone-G.
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# 1 Introduction

Generalized distributions from baseline distributions has become an important tool in statistical modeling. The generalized distributions are connected or linked to other useful distributions, which makes the derivation of useful statistical properties important. These generalized distributions produces a wide range of distributions since the baseline distribution may vary depending on the researcher's interests. Therefore, generalized distributions produces models that can be applied to a wide range of phenomena.

Available in the literature are various methods for generalizing baseline distributions by adding one or more extra parameters to the distribution. Some of the generalizations

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are the exponentiated families of distributions by Mudholkar and Srivastava (1993), Gupta et al. (1998), Nadarajah and Kotz (2006), Lemonte et al. (2013), Lemonte (2013), Huang and Oluyede (2014), and Cordeiro et al. (2013), Marshall-Olkin generated families by Marshall and Olkin (1997) and Pakungwati et al. (2018), the Kumaraswamy-generated families by Cordeiro et al. (2010), Alizadeh et al. (2015), and Chipepa et al. (2019b), the beta-generated families by Eugene et al. (2002), Jones (2004), and Chipepa et al. (2019a)), gamma-generated families by Zografos and Balakrishnan (2009), Ristić and Balakrishnan (2012), and Foya et al. (2017)), half-logistic-generated families by Cordeiro et al. (2016), Topp-Leone generated by Al-Shomrani et al. (2016) and Chipepa et al. (2020), and half-logistic families by Afify et al. (2017), Korkmaz et al. (2018a) and Cordeiro et al. (2017), to mention a few.

We were motivated by the desirable properties exhibited by the new proposed family of distributions. The new distribution (i) contains new and some well known exponentiated families of distributions; (ii) is flexibility in data fitting; (iii) exhibits a variety of shapes of the hazard rate function. We hope this new generalized family of distributions will receive much attention from both applied and theoretical statisticians.

The rest of the paper is organized as follows: In Section 2, we present the new family of distributions namely the exponentiated half logistic odd Weibull-Topp-Leone-G (EHLOW-TL-G) family of distributions and model properties. We estimate the parameters of the model in Section 3. Special cases are presented in Section 4. A simulation study is contacted in Section 5. Inference results are given in Section 6, followed by concluding remarks.

# 2 The model

In this section, a new model is proposed, namely, EHLOW-TL-G family of distributions. We also derive the statistical properties of the new family of distributions which include expansion of probability density function (pdf), quantile function, moments and generating function, entropy, distribution of order statistics and probability weighted moments.

Cordeiro et al. (2016) developed the type I half-logistic family of distributions with cumulative distribution function

$$F(x;\lambda,\zeta)=\int_0^{-\log\left(\bar{G}(x;\zeta)\right)}\frac{2\lambda e^{-\lambda t}}{(1+e^{-\lambda t})^2}dt=\frac{\bar{G}^{\lambda}(x;\zeta)}{1+\bar{G}^{\lambda}(x;\zeta)},$$

for $\lambda>0$. Also, Al-Shomrani et al. (2016) developed the Topp-Leone-G (TL-G) family of distributions with cdf given by

$$G_{TL-G}(x,\zeta)=[1-\overline{G}^2(x;\zeta)]^b, \tag{1}$$

for $b>0$ and parameter vector $\zeta$. Using the generalization by Gurvich et al. (1997), the cdf of the odd Weibull-Topp-Leone-G distribution is given by

$$F(x;b,\beta,\zeta)=1-\exp\left\{-\left[\frac{[1-\overline{G}^2(x;\zeta)]^b}{[1-(1-\overline{G}^2(x;\zeta))^b]}\right]^{\beta}\right\}. \tag{2}$$

Therefore, the cdf and pdf of EHLOW-TL-G family of distributions are given by

$$F_{EHLOW-TL-G}(x;\alpha,\beta,\delta,\zeta) = \left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta} \quad (3)$$

$$f_{EHLOW-TL-G}(x;\alpha,\beta,\delta,\zeta) = \frac{4\alpha\beta\delta g(x;\zeta)\overline{G}(x;\zeta)[1-\overline{G}^2(x;\zeta)]^{\alpha\beta-1}}{(1-[1-\overline{G}^2(x;\zeta)]^{\alpha})^{\beta+1}} \times \frac{\exp(-t)}{(1+\exp(-t))^2}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta-1}, \quad (4)$$

respectively, where $t = \left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta}$ for $\alpha,\beta,\delta > 0$ and $\zeta$ is a vector of parameters from the baseline distribution function $G(.)$.

## 2.1 Expansion of pdf

By considering the following expansion

$$(1+x)^b = \sum_{n=0}^{\infty}\binom{b}{n}x^n, \quad (5)$$

we get

$$\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta}\right)\right)^{-(\delta+1)} = \sum_{p=0}^{\infty}(-1)^p\binom{-(\delta+1)}{p} \times \exp\left(-p\left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta}\right),$$

$$\left(1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta}\right)\right)^{\delta-1} = \sum_{q=0}^{\infty}(-1)^q\binom{\delta-1}{q} \times \exp\left(-q\left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta}\right),$$

$$\exp\left(-(p+q+1)\left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta}\right) = \sum_{i=0}^{\infty}\frac{(-1)^i(p+q+1)^i}{i!} \times \frac{[1-\overline{G}^2(x;\zeta)]^{\alpha\beta i}}{[1-[1-\overline{G}^2(x;\zeta)]^{\alpha}]^{\beta i}},$$

so that (4) can be written as

$$f_{EHLOW-TL-G}(x;\alpha,\beta,\delta,\zeta) = \sum_{p,q,i=0}^{\infty}\frac{(-1)^{p+q+i}(p+q+1)^i 4\alpha\beta\delta}{i!}\binom{-(\delta+1)}{p} \times \binom{\delta-1}{q}\frac{g(x;\zeta)\overline{G}(x;\zeta)[1-\overline{G}^2(x;\zeta)]^{\alpha\beta(i+1)-1}}{(1-[1-\overline{G}^2(x;\zeta)]^{\alpha})^{\beta(i+1)+1}}.$$

Also, considering the expansions

$$(1-[1-\overline{G}^2(x;\zeta)]^{\alpha})^{-(\beta(i+1)+1)} = \sum_{m=0}^{\infty}(-1)^m\binom{-(\beta(i+1)+1)}{m}[1-\overline{G}^2(x;\zeta)]^{\alpha m},$$

$$\begin{aligned}[1-\overline{G}^2(x;\zeta)]^{\alpha\beta(i+1)+\alpha m-1} &= \sum_{n=0}^{\infty}(-1)^n\binom{\alpha\beta(i+1)+\alpha m-1}{n}\overline{G}^{2n}(x;\zeta)\\ \overline{G}^{2n+1}(x;\zeta) &= \sum_{w=0}^{\infty}(-1)^w\binom{2n+1}{w}G^w(x;\zeta),\end{aligned}$$

we have

$$\begin{aligned}f_{EHLOW-TL-G}(x;\alpha,\beta,\delta,\zeta) &= \sum_{p,q,i,m,n,w=0}^{\infty}\frac{(-1)^{p+q+i+m+n+w}(p+q+1)^i 4\alpha\beta\delta}{i!}\\ &\quad\times\binom{-(\delta+1)}{p}\binom{\delta-1}{q}\binom{-(\beta(i+1)+1)}{m}\\ &\quad\times\binom{\alpha\beta(i+1)+\alpha m-1}{n}\binom{2n+1}{w}g(x;\zeta)G^w(x;\zeta)\\ &= \sum_{w=0}^{\infty}\eta_{w+1}g_{w+1}(x;\zeta), \qquad (6)\end{aligned}$$

which is an infinite linear combination of exponentiated-G (Exp-G) distribution, where $g_{w+1}(x;\zeta) = (w+1)g(x;\zeta)G^w(x;\zeta)$ is an Exp-G distribution with power parameter $(w+1)$ and linear component

$$\begin{aligned}\eta_{w+1} &= \sum_{p,q,i,m,n=0}^{\infty}\frac{(-1)^{p+q+i+m+n+w}(p+q+1)^i 4\alpha\beta\delta}{i!(w+1)}\binom{-(\delta+1)}{p}\binom{\delta-1}{q}\\ &\quad\times\binom{-(\beta(i+1)+1)}{m}\binom{\alpha\beta(i+1)+\alpha m-1}{n}\binom{2n+1}{w}. \qquad (7)\end{aligned}$$

We can therefore, derive other statistical properties of the EHLOW-TL-G family of distributions directly from the Exp-G distribution.

## 2.2 Sub-Families

We obtain the following as sub-families from the EHLOW-TL-G family of distributions.

- When $\delta = 1$, we obtain the HLOW-TL-G family of distributions.
- When $\beta = 1$, we obtain the exponentiated Half Logistic odd exponential-Topp-Leone-G (EHLOE-TL-G) family of distributions.
- When $\beta = 2$, we obtain the exponentiated Half Logistic odd Rayleigh-Topp-Leone-G (EHLOR-TL-G) family of distributions.

- When $\alpha = 1$, we obtain the new family of distributions with cdf given by

$$F(x;\beta,\delta,\zeta) = \left[\frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]}{\overline{G}^2(x;\zeta)}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]}{\overline{G}^2(x;\zeta)}\right]^{\beta}\right)}\right]^{\delta}.$$

- When $\alpha = \beta = 1$, we obtain the new family of distributions with cdf given by

$$F(x;\delta,\zeta) = \left[\frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]}{\overline{G}^2(x;\zeta)}\right]\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]}{\overline{G}^2(x;\zeta)}\right]\right)}\right]^{\delta}.$$

- When $\alpha = \beta = \delta = 1$, we obtain the new family of distributions with cdf given by

$$F(x;\zeta) = \left[\frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]}{\overline{G}^2(x;\zeta)}\right]\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\zeta)]}{\overline{G}^2(x;\zeta)}\right]\right)}\right].$$

## 2.3 Moments and generating function

In this section, we derive the ordinary moment, central moment, incomplete moment and generating function of the EHLOW-TL-G family of distributions. We use results of series expansion presented in Section 2.1 to derive the above properties. Let $Z_{w+1}$ be an Exp-G distribution with power parameter $(w+1)$, then the $r^{th}$ ordinary moment of the EHLOW-TL-G is given by

$$\mu'_r = E(X^r) = \sum_{w=0}^{\infty} \eta_{w+1} E(Z^r_{w+1}), \tag{8}$$

where $\eta_{w+1}$ is given by (7) and $E(Z^r_{w+1})$ is the $r^{th}$ moment of the Exp-G distribution. Also, the $s^{th}$ central moment of $X$ is given by

$$\mu_s = \sum_{r=0}^{s} \binom{s}{r} (-\mu'_1)^{s-r} E(X^r) = \sum_{r=0}^{s}\sum_{w=0}^{\infty} \eta_{w+1} \binom{s}{r} (-\mu'_1)^{s-r} E(Z^r_{w+1}).$$

Furthermore, the $r^{th}$ incomplete moment of $X$ is given by

$$\phi_r(z) = \int_{-\infty}^{z} x^r f(x)dx = \sum_{w=0}^{\infty} \eta_{w+1} \int_{-\infty}^{z} x^r g_{w+1}(x;\xi)dx, \tag{9}$$

where $\int_{-\infty}^{z} x^r g_{w+1}(x;\xi)dx$ is the $r^{th}$ incomplete moment of the Exp-G distribution. The incomplete moment is used to estimate Bonferroni and Lorenz curves, which are very useful in reliability, medicine, economics, demography and insurance.

Table 1 shows the first five moments together with the standard deviation (SD or $\sigma$), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the exponentiated half logistic odd Weibull-Topp-Leone-log logistic (EHLOW-TL-LLoG) distribution for selected values of the parameters $(\alpha, \beta, \delta, \lambda)$.

Table 1: Moments of the EHLOW-TL-LLoG distribution for some parameter values

| | .5,.5,.5,1 | .5,1.5,1,1.5 | .5,1.5,1,1.5 | 1,1.5,1,.5 | 1.3,1.5,1.5,.5 |
|---|---|---|---|---|---|
| $E(X)$ | 0.1779 | 0.3043 | 0.3043 | 0.2384 | 0.3925 |
| $E(X^2)$ | 0.0974 | 0.1143 | 0.1143 | 0.0994 | 0.2283 |
| $E(X^3)$ | 0.0666 | 0.0485 | 0.0485 | 0.0539 | 0.1538 |
| $E(X^4)$ | 0.0505 | 0.0224 | 0.0224 | 0.0342 | 0.1133 |
| $E(X^5)$ | 0.0406 | 0.0111 | 0.0111 | 0.0240 | 0.0886 |
| SD | 0.2564 | 0.1474 | 0.1474 | 0.2064 | 0.2725 |
| CV | 1.4412 | 0.4844 | 0.4844 | 0.8660 | 0.6944 |
| CS | 1.5366 | 0.1456 | 0.1456 | 1.1221 | 0.2910 |
| CK | 4.3016 | 2.5268 | 2.5268 | 3.8543 | 2.1183 |

The moment generating function (mgf) of the EHLOW-TL-G family of distributions is given by

$$M_X(t) = E(e^{tX}) = \sum_{q=0}^{\infty} \eta_{w+1} M_{Z_{w+1}}(t),$$

where $M_{Z_{w+1}}(t)$ is the mgf of the Exp-G distribution.

## 2.4 Quantile function

The quantile function of the EHLOW-TL-G family of distributions is obtained by inverting the cdf as follows

$$\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta} = u$$

for $0 \leq u \leq 1$, which simplifies to

$$\log z = -\left[\frac{[1-\overline{G}^2(x;\zeta)]^{\alpha}}{1-[1-\overline{G}^2(x;\zeta)]^{\alpha}}\right]^{\beta},$$

where $z = \left[\frac{1-u^{1/\delta}}{1+u^{1/\delta}}\right]$. The equation can further be simplified to

$$\overline{G}(x;\zeta) = \left(1-\left[\frac{(-\log z)^{1/\beta}}{1+(-\log z)^{1/\beta}}\right]^{1/\alpha}\right)^{1/2},$$

which reduces to

$$G(x;\zeta) = 1-\left(1-\left[\frac{(-\log z)^{1/\beta}}{1+(-\log z)^{1/\beta}}\right]^{1/\alpha}\right)^{1/2}.$$

Hence, the quantile values of the EHLOW-TL-G family of distributions are obtained by solving the equation

$$x(u) = G^{-1}\left[1-\left(1-\left[\frac{(-\log z)^{1/\beta}}{1+(-\log z)^{1/\beta}}\right]^{1/\alpha}\right)^{1/2}\right]. \quad (10)$$

Some quantile values for some parameters values for the EHLOW-TL-LLoG distribution are shown in Table 2 for the parameters $(\alpha, \beta, \delta, \lambda)$.

Table 2: Quantile values for some parameters values for the EHLOW-TL-LLoG Distribution

| u | 1.5,1.5,1.5,1.5 | 1.5,1,1.5,1.5 | 1.5,.5,1.5,1 | 1.5,1.5,1,1.5 | 1,1.5,1,.5 |
|---|---|---|---|---|---|
| 0.1 | 0.5702 | 0.4981 | 0.1916 | 0.4417 | 0.0252 |
| 0.2 | 0.6683 | 0.6325 | 0.3923 | 0.5561 | 0.0596 |
| 0.3 | 0.7371 | 0.7329 | 0.6112 | 0.6384 | 0.0985 |
| 0.4 | 0.7941 | 0.8193 | 0.8531 | 0.7069 | 0.1419 |
| 0.5 | 0.8459 | 0.9004 | 1.1264 | 0.7685 | 0.1909 |
| 0.6 | 0.8963 | 0.9813 | 1.4461 | 0.8279 | 0.2478 |
| 0.7 | 0.9490 | 1.0679 | 1.8406 | 0.8890 | 0.3172 |
| 0.8 | 1.0095 | 1.1694 | 2.3740 | 0.9577 | 0.4098 |
| 0.9 | 1.0917 | 1.3110 | 3.2511 | 1.0491 | 0.5590 |

## 2.5 Entropy

Entropy measures variation of uncertainty of a random variable $X$, which follows a probability distribution $f(.)$. There are two common types of entropy, Rényi entropy by Rényi (1960) and Shannon entropy by Shannon (1951). Shannon entropy is a special case of Rényi entropy. In this paper, we derive the Rényi entropy $(I_R(\nu))$ of the EHLOW-TL-G family of distributions as follows

$$I_R(\nu) = (1-\nu)^{-1}\log\left[\int_0^\infty f^\nu(x)dx\right], v \neq 1, v > 0. \quad (11)$$

Using the EHLOW-TL-G pdf, $f^\nu(x)$ can be written as

$$f^\nu(x) = \frac{(4\alpha\beta\delta)^\nu g^\nu(x;\zeta)\overline{G}^\nu(x;\zeta)[1-\overline{G}^2(x;\zeta)]^{(\alpha\beta-1)\nu}}{(1-[1-\overline{G}^2(x;\zeta)]^\alpha)^{(\beta+1)\nu}} \times\frac{\exp(-\nu t)}{(1+\exp(-t))^{2\nu}}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{(\delta-1)\nu}.$$

By applying series expansions used in section 2.1, we get

$$\int_0^\infty f^\nu(x)dx = \sum_{p,q,i,m,n,w=0}^{\infty}\frac{(-1)^{p+q+i+m+n+w}(p+q+\nu)^i}{i!}\binom{-(\nu(\delta+1))}{p} \times\binom{\nu(\delta-1)}{q}(4\alpha\beta\delta)^\nu\binom{-(\beta(i+\nu)+\nu)}{m}\binom{\alpha\beta(i+\nu)+\alpha m-\nu}{n}$$

$$\times\binom{2n+\nu}{w}\frac{1}{(w/\nu+1)^{\nu}}\int_0^{\infty}\left((w/\nu+1)g^{\nu}(x;\zeta)G^{w/\nu}(x;\zeta)\right)^{\nu}dx$$

so that the Rényi entropy of the EHLOW-TL-G family of distributions is given by

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\sum_{w=0}\pi_w e^{(1-\nu)I_{REG}})\right], v\neq 1, v>0, \tag{12}$$

where

$$\begin{aligned}\pi_w &= \sum_{p,q,i,m,n=0}^{\infty}\frac{(-1)^{p+q+i+m+n+w}(p+q+\nu)^i}{i!}\binom{-(\nu(\delta+1))}{p}\binom{\nu(\delta-1)}{q}\\ &\quad\times(4\alpha\beta\delta)^{\nu}\binom{-(\beta(i+\nu)+\nu)}{m}\binom{\alpha\beta(i+\nu)+\alpha m-\nu}{n}\binom{2n+1}{w}\\ &\quad\times\frac{1}{(w/\nu+1)^{\nu}}\end{aligned}\tag{13}$$

and $I_{REG}=\int_0^{\infty}\left((w/\nu+1)g^{\nu}(x;\zeta)G^{w/\nu}(x;\zeta)\right)^{\nu}dx$ is Rényi entropy of Exp-G distribution with parameter $(w/\nu+1)$. Hence, we can directly derive the Rényi entropy of the EHLOW-TL-G family of distributions from the Rényi entropy of Exp-G distribution.

## 2.6 Distribution of order statistics

The pdf of the $i^{th}$ order statistic can be obtained using the formula given by

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}F(x)^{j+i-1}, \tag{14}$$

where $B(.,.)$ is the beta function. Substituting the cdf and pdf of the EHLOW-TL-G family of distributions into (14) and considering $f(x)F(x)^{j+i-1}$, we have

$$\begin{aligned}f(x)F(x)^{j+i-1} &= \frac{4\alpha\beta\delta g(x;\zeta)\overline{G}(x;\zeta)[1-\overline{G}^2(x;\zeta)]^{\alpha\beta-1}}{(1-[1-\overline{G}^2(x;\zeta)]^{\alpha})^{\beta+1}}\\ &\quad\times\exp(-t)\frac{(1-\exp(-t))^{\delta(j+i)-1}}{(1+\exp(-t))^{\delta(j+i)+1}}.\end{aligned}$$

Also, by applying the series expansions from section 2.1, we get

$$\begin{aligned}f(x)F(x)^{j+i-1} &= \sum_{p,q,l,m,n,w=0}^{\infty}\frac{(-1)^{p+q+l+m+n+w}(p+q+1)^l 4\alpha\beta\delta}{l!}\\ &\quad\times\binom{-(\delta(j+i)+1)}{p}\binom{\delta(j+i)-1}{q}\binom{-(\beta(l+1)+1)}{m}\\ &\quad\times\binom{\alpha\beta(l+1)+\alpha m-1}{n}\binom{2n+1}{w}g(x;\zeta)G^{w}(x;\zeta).\end{aligned}\tag{15}$$

Therefore, the distribution of the $i^{th}$ order statistic from the EHLOW-TL-G family of distributions is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{p,q,l,m,n,w=0}^{\infty}\sum_{j=0}^{n-j}\frac{(-1)^{p+q+l+m+n+w}(p+q+1)^l 4\alpha\beta\delta}{l!} \\ &\times\binom{n-i}{j}\binom{-(\delta(j+i)+1)}{p}\binom{\delta(j+i)-1}{q}\binom{-(\beta(l+1)+1)}{m} \\ &\times\binom{\alpha\beta(l+1)+\alpha m-1}{n}\binom{2n+1}{w}g(x;\zeta)G^w(x;\zeta) \\ &= \sum_{w=0}^{\infty}\eta^*_{w+1}g_{w+1}(x;\zeta), \end{aligned} \tag{16}$$

where $g_{w+1}(x;\zeta) = (w+1)g(x;\zeta)G^w(x;\zeta)$ is an Exp-G distribution with power parameter $(w+1)$ and

$$\begin{aligned} \eta^*_{w+1} &= \frac{1}{B(i,n-i+1)}\sum_{p,q,l,m,n=0}^{\infty}\sum_{j=0}^{n-j}\frac{(-1)^{p+q+l+m+n+w}(p+q+1)^l 4\alpha\beta\delta}{l!(w+1)} \\ &\times\binom{n-i}{j}\binom{-(\delta(j+i)+1)}{p}\binom{\delta(j+i)-1}{q}\binom{-(\beta(l+1)+1)}{m} \\ &\times\binom{\alpha\beta(l+1)+\alpha m-1}{n}\binom{2n+1}{w}. \end{aligned} \tag{17}$$

## 2.7 Probability weighted moments

By definition, probability weighted moments (PWMs) say $\xi_{j,i}$ of $X \sim$ EHLOW-TL-G $(\alpha,\beta,\delta,\zeta)$ distribution is given by

$$\xi_{j,i} = E(X^jF(X)^i) = \int_{-\infty}^{\infty}x^jf(x)F(x)^i dx.$$

Using (15) from the derivation of order statistics, we can write

$$\begin{aligned} f(x)F(x)^i &= \sum_{p,q,l,m,n,w=0}^{\infty}\frac{(-1)^{p+q+l+m+n+w}(p+q+1)^l 4\alpha\beta\delta}{l!} \\ &\times\binom{-(\delta(1+i)+1)}{p}\binom{\delta(1+i)-1}{q}\binom{-(\beta(l+1)+1)}{m} \\ &\times\binom{\alpha\beta(l+1)+\alpha m-1}{n}\binom{2n+1}{w}g(x;\zeta)G^w(x;\zeta), \end{aligned}$$

which simplifies to

$$f(x)F(x)^i = \sum_{w=0}^{\infty}\psi_{w+1}g_{w+1}(x;\zeta),$$

where

$$\psi_{w+1} = \sum_{p,q,l,m,n=0}^{\infty} \frac{(-1)^{p+q+l+m+n+w}(p+q+1)^l 4\alpha\beta\delta}{l!(w+1)} \times \binom{-(\delta(1+i)+1)}{p}\binom{\delta(1+i)-1}{q}\binom{-(\beta(l+1)+1)}{m} \times \binom{\alpha\beta(l+1)+\alpha m-1}{n}\binom{2n+1}{w},$$

and $g_{w+1}(x;\xi) = (w+1)g(x;\zeta)[G(x;\zeta)]^w$ is an Exp-G distribution with power parameter $(w+1)$. Therefore, the PWMs of the EHLOW-TL-G family of distributions is given by

$$\xi_{j,i} = \sum_{w=0}^{\infty} \psi_{w+1} \int_{-\infty}^{\infty} x^j g_{w+1}(x;\zeta)dx = \sum_{w=0}^{\infty} \phi_{w+1} E(Z_{w+1}^j),$$

where $Z_{w+1}^j$ is the $j^{th}$ power of an Exp-G distributed random variable with power parameter $(w+1)$.

# 3 Estimation

In this section, we obtain the Maximum Likelihood Estimators (MLE) of vector of parameters. If $X_i \sim EHLOW-TL-G(\alpha,\beta,\delta;\zeta)$ with the parameter vector $\psi = (\alpha,\beta,\delta;\zeta)^T$. The total log-likelihood $\ell = \ell(\psi)$ from a random sample of size $n$ is given by

$$\ell = n\log(4\alpha\beta) + \sum_{i=1}^{n}\log[g(x_i;\zeta)] + \sum_{i=1}^{n}\log[\overline{G}(x_i;\zeta)] + (\alpha\beta-1)\sum_{i=1}^{n}\log[1-\overline{G}^2(x_i;\zeta)] - \sum_{i=1}^{n} v_i - (\beta+1)\sum_{i=1}^{n}\log[1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}] - 2\sum_{i=1}^{n}\log[1+\exp(-v_i)] + (\delta-1)\sum_{i=1}^{n}\log\left[\frac{1-\exp(-v_i)}{1+\exp(-v_i)}\right],$$

where $v_i = \left[\frac{(1-\overline{G}^2(x_i;\zeta))^{\alpha}}{1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}}\right]^{\beta}$.

The score vector $\mathbf{U} = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\zeta_k}\right)$ has elements given by:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \beta\sum_{i=1}^{n}\log[1-\overline{G}^2(x_i;\zeta)] - \sum_{i=1}^{n}(v_i)^{-1}\frac{\beta(1-\overline{G}^2(x_i;\zeta))^{\alpha}\log(1-\overline{G}^2(x_i;\zeta))}{(1-(1-\overline{G}^2(x_i;\zeta))^{\alpha})^2} + (\beta+1)\sum_{i=1}^{n}\frac{(1-\overline{G}^2(x_i;\zeta))^{\alpha}\log(1-\overline{G}^2(x_i;\zeta))}{1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}} + 2\beta\sum_{i=1}^{n}\frac{\exp(-v_i)\log[1-\overline{G}^2(x_i;\zeta)]\big((1-\overline{G}^2(x_i;\zeta))(1-\overline{G}^2(x_i;\zeta))^{\alpha\beta-1}\big)}{\big(1+\exp(-v_i)\big)\big(1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}\big)^{\beta+1}}$$

$$
\begin{aligned}
& +(\delta-1)\sum_{i=1}^{n}\frac{2\beta\exp(-v_i)\ln[1-\overline{G}^2(x_i;\zeta)](1-\overline{G}^2(x_i;\zeta))^{\alpha\beta-1}(1-\overline{G}(x_i;\zeta))^{\alpha}}{\big(1-\exp(-v_i)\big)\big(1+\exp(-v_i)\big)(1-(1-\overline{G}(x_i;\zeta))^{\alpha})^{\beta+1}},\\
\frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta}+\alpha\sum_{i=1}^{n}\log[1-\overline{G}^2(x_i;\zeta)]-\sum_{i=1}^{n}\log[1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}]\\
& -\sum_{i=1}^{n}v_i\log\left[\frac{(1-\overline{G}^2(x_i;\zeta))^{\alpha}}{1-(1-\overline{G}^2(x_i;\zeta))^{b}}\right]\\
& +2\sum_{i=1}^{n}\frac{\exp(-v_i)\log\left[\frac{(1-\overline{G}^2(x_i;\zeta))^{\alpha}}{1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}}\right](1-\overline{G}^2(x_i;\zeta))^{\alpha\beta}}{(1+\exp(-v_i))\big(1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}\big)^{\beta}}\\
& +(\delta-1)\sum_{i=1}^{n}\frac{2v_i\exp(-v_i)\log\left(\frac{(1-\overline{G}(x_i;\zeta))^{\alpha}}{1-(1-\overline{G}(x_i;\zeta))^{\alpha}}\right)}{\big(1-\exp(-v_i)\big)\big(1+\exp(-v_i)\big)},\\
\frac{\partial\ell}{\partial\delta} &= \frac{n}{\delta}+\sum_{i=1}^{n}\frac{1-\exp(-v_i)}{1+\exp(-v_i)},\\
\frac{\partial\ell}{\partial\zeta_k} &= \sum_{i=1}^{n}\frac{1}{g(x_i;\zeta)}\frac{\partial g(x_i;\zeta)}{\partial\zeta_k}-2\sum_{i=1}^{n}\frac{1}{[\overline{G}(x_i;\zeta)]}\frac{\partial[\overline{G}(x_i;\zeta)]}{\partial\zeta_k}\\
& -(\alpha\beta-1)\sum_{i=1}^{n}\frac{1}{1-\overline{G}^2(x_i;\zeta)}\frac{\partial\big[1-\overline{G}^2(x_i;\zeta)\big]}{\partial\zeta_k}-\sum_{i=1}^{n}(v_i)^{-1}\\
& \times\frac{(1-\overline{G}^2(x;\zeta))^{\alpha}\frac{\partial[1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}]}{\partial\zeta_k}-(1-(1-\overline{G}^2(x_i;\zeta))^{\alpha})\frac{\partial(1-\overline{G}^2(x_i;\zeta))^{\alpha}}{\partial\zeta_k}}{(1-(1-\overline{G}^2(x_i;\zeta))^{\alpha})^2}\\
& -(\beta+1)\sum_{i=1}^{n}\frac{1}{1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}}\frac{\partial\big[1-(1-\overline{G}^2(x_i;\zeta))^{\alpha}\big]}{\partial\zeta_k}\\
& -2\sum_{i=1}^{n}\frac{1}{\big(1+\exp(-v_i)\big)}\frac{\partial\big(1+\exp(-v_i)\big)}{\partial\zeta_k}\\
& +(\delta-1)\sum_{i=1}^{n}\frac{1}{\left[\frac{1-\exp(-v_i)}{1+\exp(-v_i)}\right]}\frac{\partial\left[\frac{1-\exp(-v_i)}{1+\exp(-v_i)}\right]}{\partial\zeta_k},
\end{aligned}
$$

The elements of the score vector are not in closed form and can be solved using R, MATLAB and SAS software.

Furthermore, we use the observed Fisher information matrix to obtain confidence intervals for the model parameters $\psi=(\alpha,\beta,\delta,\zeta)$. The Fisher information matrix is given by

$$
J(\psi)=\begin{pmatrix}
J_{\alpha\alpha}(\psi) & J_{\alpha\beta}(\psi) & J_{\alpha\delta}(\psi) & J_{\alpha\zeta}(\psi)\\
J_{\beta\alpha}(\psi) & J_{\beta\beta}(\psi) & J_{\beta\delta}(\psi) & J_{\beta\zeta}(\psi)\\
J_{\delta\alpha}(\psi) & J_{\delta\beta}(\psi) & J_{\delta\delta}(\psi) & J_{\delta\zeta}(\psi)\\
J_{\zeta\alpha}(\psi) & J_{\zeta\beta}(\psi) & J_{\zeta\delta}(\psi) & J_{\zeta\zeta}(\psi)
\end{pmatrix}, \tag{18}
$$

where $J_{i,j} = \frac{-\partial^2 \ell(\psi)}{\partial i \partial j}$, for $i, j = \alpha, \beta, \delta, \zeta$, where $\zeta$ is a $p$ component vector, $J_{\zeta\zeta}(\psi)$ is a $p \times p$ matrix, $J_{\alpha\zeta}(\psi)$, $J_{\beta\zeta}(\psi)$ and $J_{\delta\zeta}(\psi)$ has $p \times 1$ components, respectively. Under the usual regularity conditions $\hat{\psi}$ is asymptotically normal distributed, that is $\hat{\psi} \sim N(\underline{0}, I^{-1}(\psi))$ as $n \longrightarrow \infty$, where $I(\psi)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\psi)$ is replaced by $J(\hat{\psi})$, the information matrix evaluated at $\hat{\psi}$.

# 4 Some special cases

In this section, we present five special cases of the EHLOW-TL-G family of distributions. We considered cases when the baseline distribution is power function, Burr XII, uniform, Kumaraswamy and Weibull distributions.

## 4.1 Exponentiated half-logistic odd Weibull-Topp-Leone-power distribution

Consider the power distribution as the baseline distribution with pdf and cdf given by $g(x; \lambda, c) = c\lambda^c x^{c-1}$ and $G(x; \lambda, c) = (\lambda x)^c$, for $0 < x < 1/\lambda$ and $c > 0$, respectively. The cdf and pdf of the exponentiated half logistic odd Weibull-Topp-Leone-power (EHLOW-TL-P) distribution are given by

$$
\begin{aligned}
F_{EHLOW-TL-P}(x; \alpha, \beta, \delta, \lambda, c) &= \left[\frac{1 - \exp(-p)}{1 + \exp(-p)}\right]^{\delta} \\
f_{EHLOW-TL-P}(x; \alpha, \beta, \delta, \lambda, c) &= \frac{4\alpha\beta\delta c\lambda^c x^{c-1}(1 - (\lambda x)^c)[1 - (1 - (\lambda x)^c)^2]^{\alpha\beta-1}}{(1 - [1 - (1 - (\lambda x)^c)^2]^{\alpha})^{\beta+1}} \\
&\quad \times \exp(-p)\big(1 + \exp(-p)\big)^{-2}\left[\frac{1 - \exp(-p)}{1 + \exp\big(-p\big)}\right]^{\delta-1},
\end{aligned}
$$

respectively, where $p = \left[\frac{[1-(1-(\lambda x)^c)^2]^{\alpha}}{1-[1-(1-(\lambda x)^c)^2]^{\alpha}}\right]^{\beta}$, $\alpha, \beta, \delta, c > 0$ and $0 < x < 1/\lambda$.

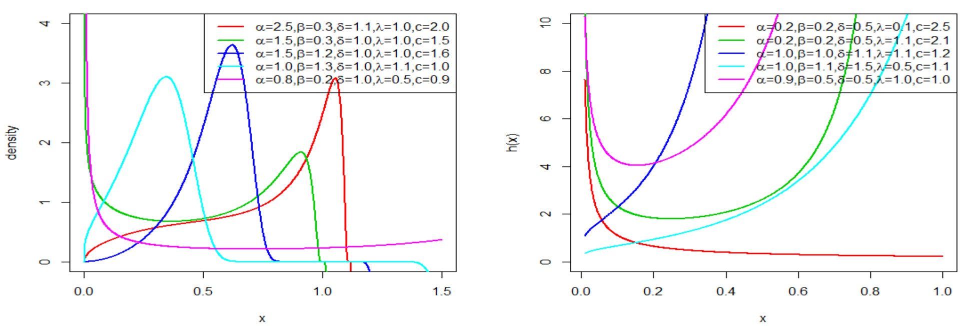


Figure 1: Plots of the pdf and hrf for the EHLOW-TL-P distribution

Figure 1 shows the pdfs and hazard functions for the EHLOW-TL-P distribution. The pdfs exhibits various shapes that include reverse-J, left and right-skewed. The distribution also has platykurtic pdfs. The distribution has flexible hazard rate function that takes both non-monotonic and monotonic shapes.

## 4.2 Exponentiated half-logistic odd Weibull-Topp-Leone-Burr XII distribution

Consider the Burr XII distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda,\gamma) = \lambda\gamma x^{\lambda-1}(1+x^{\lambda})^{-\gamma-1}$ and $G(x;\lambda,\gamma) = 1-(1+x^{\lambda})^{-\gamma}$, respectively, for $c, k > 0$. The cdf and pdf of the exponentiated half logistic odd Weibull-Topp-Leone-Burr XII (EHLOW-TL-BXII) distribution are given by

$$\begin{aligned}
F_{EHLOW-TL-BXII}(x;\alpha,\beta,\delta,\lambda,\gamma) &= \left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta} \\
f_{EHLOW-TL-BXII}(x;\alpha,\beta,\delta,\lambda,\gamma) &= \frac{4\alpha\beta\delta\lambda\gamma x^{\lambda-1}(1+x^{\lambda})^{-2\gamma-1}[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha\beta-1}}{(1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha})^{\beta+1}} \\
&\quad \times \exp(-t)\big(1+\exp(-t)\big)^{-2}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta-1},
\end{aligned}$$

respectively, where $t = \left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}$, $\alpha,\beta,\delta,\lambda,\gamma > 0$. By letting $\lambda = 1$ and $\gamma = 1$, we obtain the exponentiated half logistic odd Weibull-Topp-Leone-Lomax (EHLOW-TL-Lx) and the exponentiated half logistic odd Weibull-Topp-Leone-log logistic (EHLOW-TL-LLoG) distributions, respectively from the EHLOW-TL-BXII distribution.

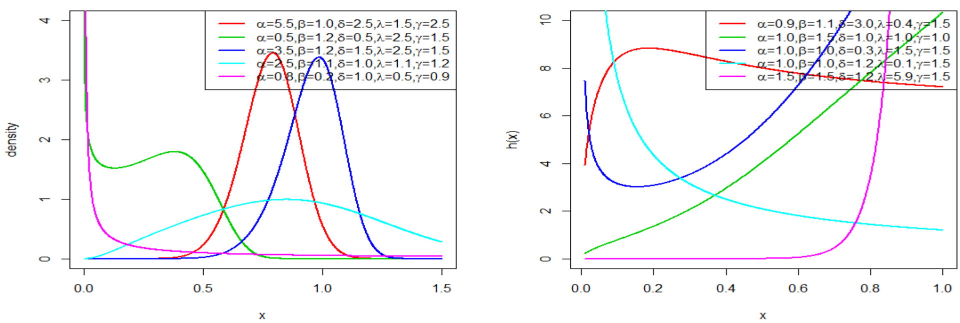


Figure 2: Plots of the pdf and hrf for the EHLOW-TL-BXII distribution

Figure 2 shows the pdfs and hazard functions for the EHLOW-TL-BXII distribution. The distribution has a very flexible hazard rate function that exhibit bathtub, upside bathtub, decreasing and increasing shapes. Also, the model is versatile in data fitting since the pdfs exhibit different levels of kurtosis and skewness.

## 4.3 Exponentiated half-logistic odd Weibull-Topp-Leone- uniform distribution

If we take the uniform distribution as the uniform distribution with pdf and cdf given by $g(x;\lambda) = 1/\lambda$ and $G(x;\lambda) = x/\lambda$, respectively, for $0 < x < \lambda$, we obtain the exponentiated half logistic odd Weibull-Topp-Leone-uniform (EHLOW-TL-U) distribution with cdf and pdf given by

$$\begin{aligned} F_{EHLOW-TL-U}(x;\alpha,\beta,\delta,\lambda) &= \left[\frac{1-\exp(-r)}{1+\exp(-r)}\right]^{\delta} \\ f_{EHLOW-TL-U}(x;\alpha,\beta,\delta,\lambda) &= \frac{4\alpha\beta\delta(1-x/\lambda)[1-(1-x/\lambda)^2]^{\alpha\beta-1}}{\lambda(1-[1-(1-x/\lambda)^2]^{\alpha})^{\beta+1}} \\ &\quad \times \exp(-r)\big(1+\exp(-r)\big)^{-2}\left[\frac{1-\exp(-r)}{1+\exp(-r)}\right]^{\delta-1} \end{aligned}$$

respectively, where $r = \left[\frac{[1-(1-x/\lambda)^2]^{\alpha}}{1-[1-(1-x/\lambda)^2]^{\alpha}}\right]^{\beta}$, $\alpha,\beta,\delta > 0$ and $0 < x < \lambda$.

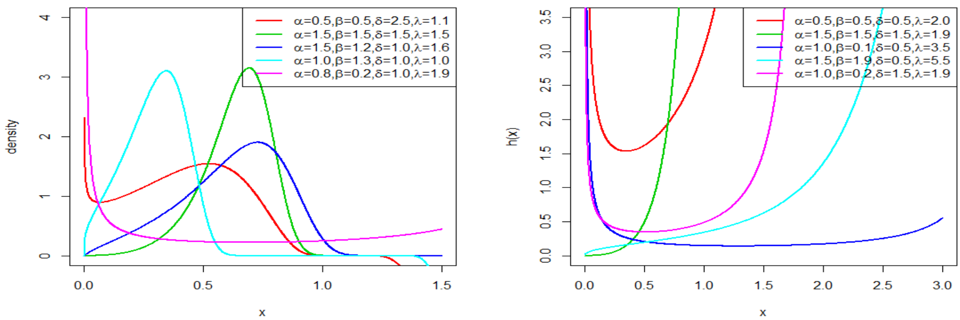


Figure 3: Plots of the pdf and hrf for the EHLOW-TL-U distribution

Figure 3 shows the pdfs and hazard functions for the EHLOW-TL-U distribution. The pdfs exhibit various shapes that include reverse-J, left and right skewed. The distribution also addresses variation in kurtosis. The hazard rate function (hrf) also exhibit bathtub, increasing, J and reverse-J shapes.

## 4.4 Exponentiated half-logistic odd Weibull-Topp-Leone- Kumaraswamy distribution

By considering the Kumaraswamy distribution as the baseline distribution with pdf and cdf given by $g(x;a,b) = abx^{a-1}(1-x^a)^{b-1}$ and $G(x;a,b) = 1-(1-x^a)^b$, for $a,b > 0$, respectively, we get the exponentiated Half logistic odd Weibull-Topp-Leone-Kumaraswamy (EHLOW-TL-Kw) distribution with cdf and pdf given by

$$F_{EHLOW-TL-Kw}(x;\alpha,\beta,\delta,a,b) = \left[\frac{1-\exp(-w)}{1+\exp(-w)}\right]^{\delta}$$

$$f_{EHLOW-TL-Kw}(x;\alpha,\beta,\delta,a,b) = \frac{4\alpha\beta\delta abx^{a-1}(1-x^a)^{2b-1}[1-(1-x^a)^{2b}]^{\alpha\beta-1}}{(1-[1-(1-x^a)^{2b}]^\alpha)^{\beta+1}} \times \exp(-w)\big(1+\exp(-w)\big)^{-2}\left[\frac{1-\exp(-w)}{1+\exp(-w)}\right]^{\delta-1},$$

respectively, where $w = \left[\frac{[1-(1-x^a)^{2b}]^\alpha}{1-[1-(1-x^a)^{2b}]^\alpha}\right]^\beta$, $\alpha,\beta,\delta,a,b>0$.

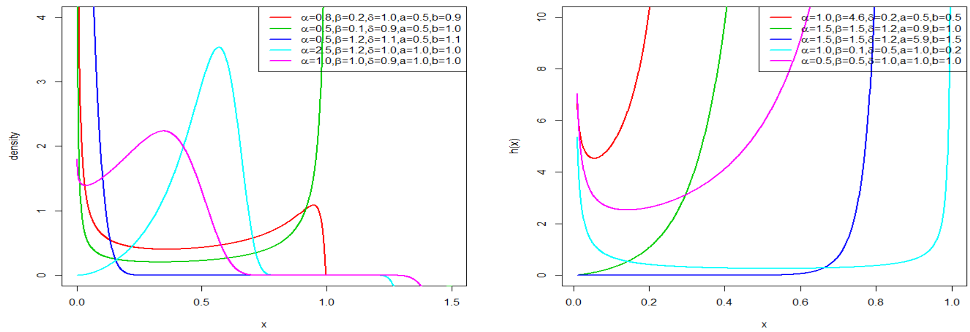


Figure 4: Plots of the pdf and hrf for the EHLOW-TL-Kw distribution

Figure 4 shows the pdfs and hazard functions for the EHLOW-TL-Kw distribution. The distribution is more versatile in data fitting because the pdf exhibit various shapes that include almost symmetric, reverse-J, right-skewed and U shape. The distribution can also be applied to data sets that has monotonic or non-monotonic hazard rate functions.

### 4.5 Exponentiated half-logistic odd Weibull-Topp-Leone-Weibull distribution

Taking the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x;\theta,\gamma) = \theta\gamma x^{\gamma-1}e^{-\theta x^\gamma}$ and $G(x;\theta,\gamma) = 1-e^{-\theta x^\gamma}$, for $\theta,\gamma>0$, respectively, we get the exponentiated half logistic odd Weibull-Topp-Leone-Weibull (EHLOW-TL-W) distribution are given by

$$F_{EHLOW-TL-W}(x;\alpha,\beta,\delta,\theta,\gamma) = \left[\frac{1-\exp(-q)}{1+\exp(-q)}\right]^\delta$$

$$f_{EHLOW-TL-W}(x;\alpha,\beta,\delta,\theta,\gamma) = \frac{4\alpha\beta\delta\theta\gamma x^{\gamma-1}e^{-2\theta x^\gamma}[1-e^{-2\theta x^\gamma}]^{\alpha\beta-1}}{(1-[1-e^{-2\theta x^\gamma}]^\alpha)^{\beta+1}} \times \exp(-q)\big(1+\exp(-q)\big)^{-2}\left[\frac{1-\exp(-q)}{1+\exp(-q)}\right]^{\delta-1},$$

respectively, where $q = \left[\frac{[1-e^{-2\theta x^\gamma}]^\alpha}{1-[1-e^{-2\theta x^\gamma}]^\alpha}\right]^\beta$, $\alpha,\beta,\delta,\theta,\gamma>0$.

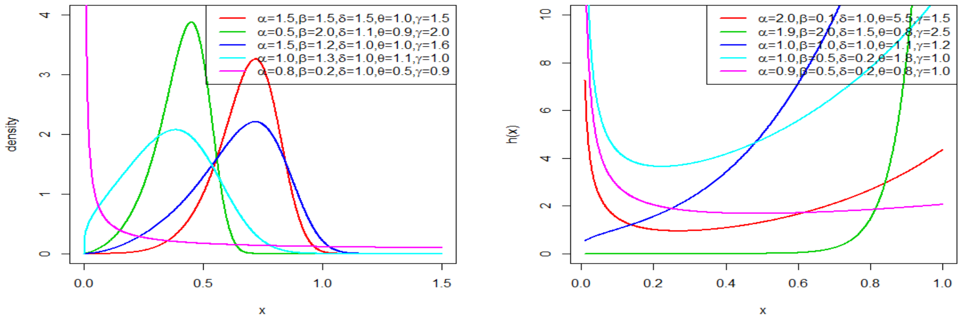


Figure 5: Plots of the pdf and hrf for the EHLOW-TL-W distribution

Figure 5 shows the pdfs and hazard functions for the EHLOW-TL-W distribution. The pdfs take various shapes that include reverse-J, left or right-skewed and almost symmetric. The distribution also addresses variation in kurtosis. The hrf also exhibit bathtub, increasing, decreasing and J shapes.

## 5 Simulation Study

A simulation study is conducted to evaluate consistency of the maximum likelihood estimates. We used R statistical software through the package (stats4), command MLE, for the EHLOW-TL-LLoG distribution. We simulated for sample sizes $n$=25, 50, 100, 200, 400, 800 and 1000 for N=1000 from the EHLOW-TL-LLoG distribution for the following sets of parameters values: $I: \ \alpha = 1, \beta = 1, \delta = 1, \lambda = 1$, $II: \ \alpha = 1.2, \beta = 1.2, \delta = 0.5, \lambda = 1$ and $III: \ \alpha = 1.2, \beta = 1.2, \delta = 1.0, \lambda = 1$. The simulation results are presented in Table 3. We assessed performance of the MLE using the mean, root mean square error (RMSE) and average bias. We expect the values of the RMSE and bias to decay toward zero for increased sample sizes if the MLE are consistent. As the sample size increases, the mean approximates the true parameter values. Also, as the sample size increases, the RMSEs and average bias decays toward zero for all the parameters.

## 6 Inference

In this section, we present real data examples to demonstrate the usefulness of the EHLOW-TL-LLoG distribution. Various goodness-of-fit statistics were used to assess model performance and these include -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer-von Mises ($W^*$) and Andersen-Darling ($A^*$) (see Chen and Balakrishnan (1995) for details), sum of squares (SS), Kolmogorov-Smirnov (K-S) statistic and its P-value. The model with the smaller values of these goodness-of-fit statistics and bigger p-values of K-S statistics is regarded as the best model.

Table 3: Monte Carlo simulation results for EHLOW-TL-LLoG distribution: Mean, RMSE and average bias when $\lambda = 1$.

| | | $\alpha=1,\beta=1,\delta=1$ | | | $\alpha=1.2,\beta=1.2,\delta=0.5$ | | | $\alpha=1.2,\beta=1.2,\delta=1.0$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $n$ | Mean | RMSE | Bias | Mean | RMSE | Bias | Mean | RMSE | Bias |
| $\alpha$ | 25 | 1.097 | 0.736 | 0.097 | 1.090 | 0.704 | -0.109 | 1.189 | 0.736 | -0.010 |
| | 50 | 1.151 | 0.719 | 0.151 | 1.219 | 0.660 | 0.019 | 1.197 | 0.685 | -0.002 |
| | 100 | 1.126 | 0.670 | 0.126 | 1.228 | 0.596 | 0.028 | 1.206 | 0.663 | 0.006 |
| | 200 | 1.179 | 0.619 | 0.179 | 1.310 | 0.525 | 0.110 | 1.269 | 0.589 | 0.069 |
| | 400 | 1.159 | 0.571 | 0.159 | 1.367 | 0.479 | 0.167 | 1.248 | 0.549 | 0.048 |
| | 800 | 1.115 | 0.485 | 0.115 | 1.397 | 0.441 | 0.197 | 1.248 | 0.489 | 0.048 |
| | 1000 | 1.106 | 0.457 | 0.106 | 1.394 | 0.433 | 0.194 | 1.243 | 0.468 | 0.043 |
| $\beta$ | 25 | 1.847 | 2.936 | 0.847 | 1.748 | 1.367 | 0.548 | 2.133 | 3.165 | 0.933 |
| | 50 | 1.629 | 1.287 | 0.629 | 1.800 | 1.255 | 0.600 | 1.702 | 1.250 | 0.502 |
| | 100 | 1.484 | 1.064 | 0.484 | 1.692 | 1.102 | 0.492 | 1.628 | 1.150 | 0.428 |
| | 200 | 1.502 | 1.027 | 0.502 | 1.720 | 1.056 | 0.520 | 1.655 | 1.037 | 0.455 |
| | 400 | 1.461 | 0.965 | 0.461 | 1.759 | 1.046 | 0.559 | 1.614 | 0.983 | 0.414 |
| | 800 | 1.333 | 0.798 | 0.333 | 1.790 | 1.105 | 0.590 | 1.555 | 0.879 | 0.355 |
| | 1000 | 1.301 | 0.746 | 0.301 | 1.776 | 1.058 | 0.576 | 1.538 | 0.871 | 0.338 |
| $\delta$ | 25 | 3.084 | 6.077 | 2.084 | 1.603 | 2.449 | 1.103 | 4.589 | 11.565 | 3.589 |
| | 50 | 2.039 | 2.565 | 1.039 | 1.074 | 1.400 | 0.574 | 2.286 | 2.962 | 1.286 |
| | 100 | 1.766 | 2.114 | 0.766 | 0.812 | 0.856 | 0.312 | 1.953 | 2.475 | 0.953 |
| | 200 | 1.392 | 1.519 | 0.392 | 0.606 | 0.430 | 0.106 | 1.524 | 1.758 | 0.524 |
| | 400 | 1.221 | 0.936 | 0.221 | 0.526 | 0.214 | 0.026 | 1.352 | 1.183 | 0.352 |
| | 800 | 1.113 | 0.558 | 0.113 | 0.489 | 0.135 | -0.010 | 1.195 | 0.725 | 0.195 |
| | 1000 | 1.086 | 0.455 | 0.086 | 0.492 | 0.129 | -0.007 | 1.160 | 0.590 | 0.160 |
| c | 25 | 1.171 | 1.128 | 0.171 | 1.145 | 0.948 | 0.145 | 1.130 | 1.025 | 0.130 |
| | 50 | 1.032 | 0.790 | 0.032 | 0.983 | 0.718 | -0.016 | 1.122 | 0.882 | 0.122 |
| | 100 | 0.993 | 0.613 | -0.006 | 0.997 | 0.595 | -0.002 | 1.082 | 0.672 | 0.082 |
| | 200 | 0.930 | 0.492 | -0.069 | 0.940 | 0.498 | -0.059 | 0.998 | 0.552 | -0.001 |
| | 400 | 0.894 | 0.413 | -0.105 | 0.873 | 0.412 | -0.126 | 0.964 | 0.473 | -0.035 |
| | 800 | 0.919 | 0.350 | -0.080 | 0.845 | 0.385 | -0.154 | 0.955 | 0.417 | -0.044 |
| | 1000 | 0.919 | 0.328 | -0.080 | 0.833 | 0.372 | -0.166 | 0.954 | 0.398 | -0.045 |

Model parameters were estimated using the MLE technique using the nlm package in R software. We present parameters estimates (standard errors in parentheses) and goodness-of-fit statistics in Tables 4 and 5. We also provide plots of fitted densities and probability plots as suggested by Chambers et al. (1983) to demonstrate how the EHLOW-TL-LLoG model fit the real data sets compared to the other several non-nested models.

The EHLOW-TL-LLoG distribution was compared to several non-nested models and these are the Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) by Chipepa et al. (2019b), Kumaraswamy-Weibull (KwW) by Cordeiro et al. (2010), beta-Weibull (BW) by Cordeiro et al. (2013), beta odd Lindley-exponential (BOL-E) and beta odd Lindley-uniform by Chipepa et al. (2019a), the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. (2018), Topp-Leone-Weibull-Lomax (TL-WLx) by Jamal et al. (2019), and Topp-Leone-Marshall-Olkin-Weibull (TLMO-W) by Chipepa et al. (2020). The pdfs of the non-nested models are as follows:

$$f_{KOL-LLoG}(x;a,b,\lambda,c) = ab\left[\frac{\lambda^2}{(1+\lambda)}\frac{cx^{c-1}}{(1+x^c)^{-1}}\exp(-\lambda z)\right] \times\left[1-\frac{\lambda+((1+x^c)^{-1})}{(1+\lambda)((1+x^c)^{-1})}\exp(-\lambda z)\right]^{a-1}$$

$$\times\left(1-\left[1-\frac{\lambda+((1+x^{c})^{-1})}{(1+\lambda)((1+x^{c})^{-1})}\exp(-\lambda z)\right]^{a}\right)^{b-1},$$

where $z=\frac{(1-(1+x^{c})^{-1})}{((1+x^{c})^{-1})}$, $a,b,\lambda,c>0$,

$$f_{BW}(x;a,b,\alpha,\beta)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}e^{-b(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1},$$

for $a,b,\alpha,\beta>0$,

$$f_{KwW}(x;a,b,\alpha,\beta)=ab\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1}(1-(1-e^{-(\alpha x)^{\beta}})^{a})^{b-1},$$

for $a,b,\alpha,\beta>0$,

$$\begin{aligned}f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda) &= \frac{\alpha\theta^{2}\gamma\lambda^{\gamma}x^{\gamma-1}e^{-(\lambda x)^{\gamma}}(e^{-(\lambda x)})^{\alpha\theta-1}(1-e^{-(\lambda x)^{\gamma}})^{\alpha-1}}{(\theta+\beta)((1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}})^{\theta-1}}\\ &\quad\times\left(1-\beta\log\left[\frac{e^{-(\lambda x)^{\gamma}}}{(1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}}}\right]\right),\end{aligned}$$

for $\alpha,\beta,\gamma,\theta,\lambda>0$,

$$\begin{aligned}f_{TL-WLx}(x;a,b,\alpha,\theta) &= 2\theta\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1}\\ &\quad\times\exp\left(-2\Big(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\Big)\right)\\ &\quad\times\left[1-\exp\left(-2\Big(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\Big)\right)\right]^{\theta-1},\end{aligned}$$

for $a,b,\alpha,\theta>0$,

$$\begin{aligned}f_{BOL-U}(x;a,b,\lambda,\theta) &= \frac{1}{B(a,b)}\left[1-\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{a-1}\\ &\quad\times\left[\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{b-1}\\ &\quad\times\frac{\lambda^{2}}{(1+\lambda)}\frac{\theta^{2}}{(\theta-x)^{3}}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\},\end{aligned}$$

for $a,b,\lambda,\theta>0$,

$$\begin{aligned}f_{BOL-E}(x;a,b,\lambda,\theta) &= \frac{1}{B(a,b)}\left[1-\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{(1-e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{a-1}\\ &\quad\times\left[\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{(1-e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{b-1}\\ &\quad\times\frac{\lambda^{2}}{(1+\lambda)}\frac{(\theta e^{-\theta x})}{e^{-3\theta x}}\exp\left\{-\lambda\frac{1-e^{-\theta x}}{e^{-\theta x}}\right\},\end{aligned}$$

for $a,b,\lambda,\theta>0$, and

$$f_{TLMO-W}(x;b,\delta,\lambda,\gamma)=\frac{2b\delta^{2}\lambda\gamma x^{\gamma-1}e^{-2\lambda x^{\gamma}}}{(1-\bar{\delta}e^{-\lambda x^{\gamma}})^{3}}\left[1-\frac{\delta^{2}e^{-2\lambda x^{\gamma}}}{(1-\bar{\delta}e^{-\lambda x^{\gamma}})^{2}}\right]^{b-1},$$

for $b,\delta,\lambda,\omega>0$. For the ELOLLOW distribution we considered the case when $\alpha=1$.

## 6.1 Chemotherapy data

We first consider the data set reported by Bekker and Mostert (1998) which represent the survival times (in years) of patients given chemotherapy treatment alone. The 46 observations are as follows: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Table 4: Parameter estimates and goodness-of-fit statistics for various models fitted for chemotherapy data set

| Model | Estimates | | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $\beta$ | $\delta$ | $\lambda$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | $KS$ | $P-value$ |
| EHLOW-TL-LLoG | 9.2518 | 1.2072 | 0.1640 | 0.9295 | 111.4 | 119.4 | 120.4 | 126.6 | 0.0401 | 0.2968 | 0.0810 | 0.9063 |
| | (4.7098) | (0.6947) | (0.1385) | (0.1767) | | | | | | | | |
| | a | b | $\lambda$ | c | | | | | | | | |
| KOL-LLoG | 2.4478 | 0.1302 | 7.4117 | 0.8769 | 114.9 | 122.9 | 123.9 | 130.2 | 0.0541 | 0.3830 | 0.0969 | 0.7562 |
| | (0.8768) | (0.1585) | (7.9891) | (0.2028) | | | | | | | | |
| | b | $\delta$ | $\lambda$ | $\gamma$ | | | | | | | | |
| TLMO-W | 1.5663 | 0.8611 | 0.4921 | 0.8424 | 116.1 | 124.1 | 125.1 | 131.3 | 0.0682 | 0.4650 | 0.0991 | 0.7320 |
| | (1.9293) | (1.0623) | (0.7341) | (0.6036) | | | | | | | | |
| | a | b | $\lambda$ | $\theta$ | | | | | | | | |
| BOL-U | 0.9574 | 3.3672 | 3.8385 | 17.1583 | 116.0 | 124.0 | 125.0 | 131.1 | 0.0936 | 0.6198 | 0.1196 | 0.5027 |
| | (0.3187) | (7.7617) | (6.8110) | (30.4900) | | | | | | | | |
| BOL-E | 1.0442 | 0.3782 | 22.3842 | 0.0829 | 116.0 | 124.0 | 125.0 | 131.2 | 0.0860 | 0.5723 | 0.1230 | 0.4677 |
| | (0.2623) | (0.4825) | (27.4203) | (0.0804) | | | | | | | | |
| | $a$ | $b$ | $\lambda$ | $k$ | | | | | | | | |
| BW | 2.0869 | 11.1746 | 0.0683 | 0.6816 | 116.0 | 124.0 | 125.0 | 131.2 | 0.0662 | 0.4538 | 0.0982 | 0.7411 |
| | (3.6003) | (32.5966) | (0.2788) | (0.6711) | | | | | | | | |
| | $a$ | $b$ | $\alpha$ | $\beta$ | | | | | | | | |
| KwW | 10.5120 | 157.1692 | 0.5989 | 0.1637 | 116.0 | 124.0 | 125.0 | 131.3 | 0.0688 | 0.4686 | 0.1021 | 0.6979 |
| | (4.9909) | (0.0679) | (1.3694) | (0.0464) | | | | | | | | |
| | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ | | | | | | | | |
| ELOLLW | 0.3336 | 0.3242 | 2.6099 | 1.0468 | 116.2 | 124.2 | 125.2 | 131.5 | 0.0819 | 0.5471 | 0.1098 | 0.6104 |
| | (1.8652) | (0.1816) | (0.2177) | (0.1389) | | | | | | | | |
| | a | b | $\alpha$ | $\theta$ | | | | | | | | |
| TL-WLx | 0.1876 | 9.1119 | 8.7808 | 0.1567 | 113.3 | 121.3 | 122.3 | 128.6 | 0.0836 | 0.5588 | 0.1134 | 0.5700 |
| | (0.0945) | (17.7419) | (9.1243) | (0.2051) | | | | | | | | |

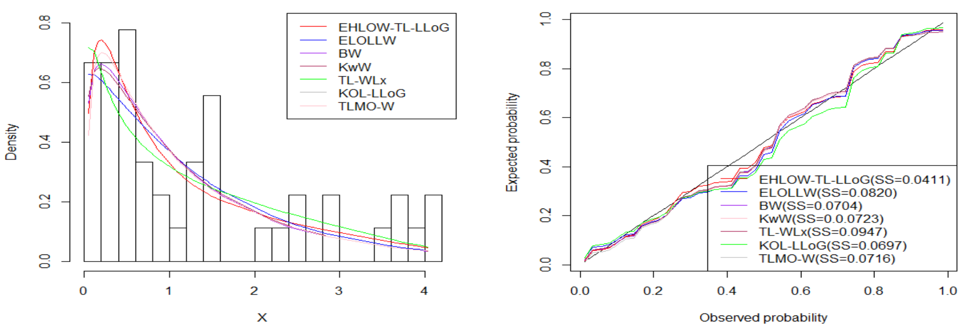


Figure 6: Fitted densities and probability plots for chemotherapy data

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 22.1818 & 1.2737 & -0.5697 & 0.3671 \\ 1.2737 & 0.4826 & -0.0726 & -0.0673 \\ -0.5697 & -0.0726 & 0.0191 & -0.0014 \\ 0.3671 & -0.0673 & -0.0014 & 0.0312 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$$\alpha \in [9.2518 \pm 9.2311], \beta \in [1.2072 \pm 1.3616],$$
$$\delta \in [0.1640 \pm 0.2715], c \in [0.9295 \pm 0.3464].$$

Based on the results shown in Table 4, we observe that the EHLOW-TL-LLoG model has the smallest values of all the goodness-of-fit statistics and bigger value for the P-value of the K-S statistic. We therefore conclude that the EHLOW-TL-LLoG distribution fit the survival times of patients on chemotherapy alone better than the several non-nested models considered in this paper. The fitted pdf to histogram also show that the EHLOW-TL-LLoG model fit the data set better than the selected non-nested models.

## 6.2 Failure times data

We considered as the second example, the data set reported by Murthy et al. (2004) which represent failure times (per 1000h) of 50 components. The observations are as follows: 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08.

Table 5: Parameter estimates and goodness-of-fit statistics for various models fitted for failure times data set

| | Estimates | | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $\alpha$ | $\beta$ | $\delta$ | $\lambda$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | $KS$ | $P-value$ |
| EHLOW-TL-LLoG | 3.3865 | 0.2229 | 0.9063 | 1.3145 | 200.1 | 208.1 | 209.0 | 215.7 | 0.1110 | 0.7179 | 0.1062 | 0.5889 |
| | (1.5417) | (0.1836) | (0.3016) | (0.9167) | | | | | | | | |
| | a | b | $\lambda$ | c | | | | | | | | |
| KOL-LLoG | 7.8627 | 0.1240 | 7.8882 | 0.4448 | 201.9 | 209.9 | 210.8 | 217.6 | 0.1490 | 0.9187 | 0.1336 | 0.3063 |
| | (50.8505) | (0.2166) | (17.1901) | (0.2814) | | | | | | | | |
| | b | $\delta$ | $\lambda$ | $\gamma$ | | | | | | | | |
| TLMO-W | 0.6761 | 1.3029 | 0.1886 | 0.8232 | 204.7 | 212.7 | 213.5 | 220.3 | 0.1469 | 0.9372 | 0.1403 | 0.2538 |
| | (1.2480) | (1.4028) | (0.5680) | (1.0324) | | | | | | | | |
| | a | b | $\lambda$ | $\theta$ | | | | | | | | |
| BOL-U | 0.4803 | 1.9720 | 3.3677 | 41.6212 | 204.0 | 212.0 | 212.8 | 219.6 | 0.1390 | 0.9233 | 0.1506 | 0.1870 |
| | (0.1014) | (7.0742) | (10.9295) | (34.2710) | | | | | | | | |
| BOL-E | 0.4939 | 1.2329 | 3.9341 | 0.0373 | 204.2 | 212.2 | 213.1 | 219.9 | 0.1409 | 0.9254 | 0.1491 | 0.1954 |
| | (0.1032) | (3.8386) | (13.2498) | (0.0452) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\gamma$ | $\theta$ | | | | | | | | |
| | $a$ | $b$ | $\lambda$ | $k$ | | | | | | | | |
| BW | 1.4514 | 9.9529 | 0.0111 | 0.5250 | 204.7 | 212.7 | 213.6 | 220.3 | 0.1549 | 0.9657 | 0.1188 | 0.4460 |
| | (3.4572) | (0.0651) | (0.0052) | (0.7545) | | | | | | | | |
| | $a$ | $b$ | $\alpha$ | $\beta$ | | | | | | | | |
| KwW | 3.6048 | 110.40 | 0.0017 | 0.2114 | 204.8 | 212.8 | 213.7 | 220.5 | 0.1555 | 0.9699 | 0.1208 | 0.4253 |
| | (0.0010) | ($2.1385 \times 10^{-5}$) | (0.0010) | (0.0211) | | | | | | | | |
| | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ | | | | | | | | |
| ELOLLW | 1.1051 | 0.0423 | 5.0895 | 0.6521 | 204.7 | 212.7 | 213.6 | 220.3 | 0.1512 | 0.9494 | 0.1285 | 0.3512 |
| | (3.0861) | (0.0295) | (0.6697) | (0.0861) | | | | | | | | |
| | a | b | $\alpha$ | $\theta$ | | | | | | | | |
| TL-WLx | 0.3141 | 0.5138 | 4.8158 | 0.1075 | 202.5 | 210.5 | 211.4 | 218.1 | 0.1247 | 0.8344 | 0.1435 | 0.2317 |
| | (0.1369) | (0.5271) | (4.1670) | (0.0939) | | | | | | | | |

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 2.3769 & 0.1584 & -0.4160 & -0.7945 \\ 0.1584 & 0.0337 & -0.0315 & -0.1671 \\ -0.4160 & -0.0315 & 0.0909 & 0.1540 \\ -0.7945 & -0.1671 & 0.1540 & 0.8403 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$$\alpha \in [3.3865 \pm 3.0218], \beta \in [0.2229 \pm 0.3599],$$

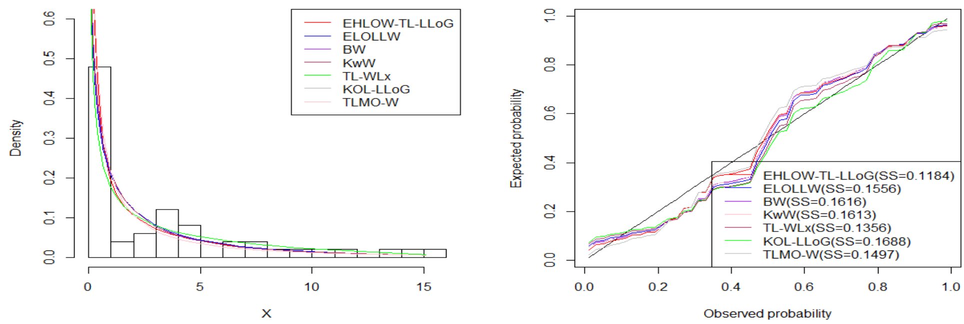


Figure 7: Fitted densities and probability plots for failure times data

$$\delta \in [0.9063 \pm 0.5911], c \in [1.3145 \pm 1.7967].$$

We also deduce from the second example that the EHLOW-TL-LLoG distribution performs better than the several selected non-nested models considered in this paper. The EHLOW-TL-LLoG model has smaller values for the goodness-of-fit statistics and bigger value for the p-value of the K-S statistic as shown in Table 5.

# 7 Conclusions

We developed a new family of distributions referred to as the exponentiated half logistic odd Weibull-Topp-Leone-G (EHLOW-TL-G) family of distributions. Statistical properties of the proposed family of distributions are also presented. Maximum likelihood estimates for the model parameters were also derived followed by a simulation study to evaluate consistency of the maximum likelihood estimates. We applied the EHLOW-TL-LLoG distribution to two real data examples. From the applications presented, we conclude that the EHLOW-TL-LLoG model performs better than several non-nested models considered in this paper.

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# Appendix

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