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Discrete degradation modeling using generalized mixed effect model

Fatemeh Hassantabar Darzi, Hasan Misaii*,
Zahra Rezaei Ghahroodi, Firoozeh Haghighi
Department of Statistics, University of Tehran, Tehran, Iran

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Abstract: Degradation modeling is an effective approach for reliability assessment and predicting of remaining useful life in order to investigate the relation between failure time and degradation of a system or unit. The possibility of an association between degradation of the unit during the time and the effects of covariates on degradation processes should be taken into account in the model to improve the explanatory capabilities of degradation models. Sometimes, the exact amount of degradation could not be observed because of time, cost, and measurement tools limitations. Therefore, approximate degradation values can be compared with a critical threshold. In this paper, the degradation processes modeled with a generalized linear mixed-effect model in order to take into account the correlation between times. Also, maximum likelihood estimation and Bayesian estimate of parameters are derived.

Keywords: Generalized linear mixed effect models; Degradation critical threshold; Degradation true path; Degradation Bayesian modeling.
Mathematics Subject Classification (2010): 62C10

1 Introduction

Today, many development devices are designed in while they work normally without failure for years. Few units will fail during the traditional reliability tests. Therefore, it becomes difficult to use a traditional life test, which only uses failure-times to assess a product's reliability. In recent years, degradation analysis has become common and important, increasingly.

*Corresponding author: hmisaii@ut.ac.ir

Reliability analysis for highly reliable components or units could be more effective using degradation modeling. In reliability theory, the failure of products includes degradation failure and sudden failure. The degradation data could provide more information than the traditional truncated failure time data. The degradation process could be effective in finding the physical model between product degradation and the accelerated stress. The degradation process also, provides direct modeling of failure mechanism which more reliable, accurate estimates and a better foundation for reliability statistical inference, see Jiang et al. (2017) and Hu et al. (2017). Lu and Meeker (1993) improves the reliability inference over the standard failure time analysis by degradation analysis. They also describe failure mechanisms and obtain reliability information of products in a short test period.

Typically, two types of degradation data are considered, physical degradation and performance degradation which are functions of time. For analyzing the general and accelerated degradation test (ADT), both the ML method and Bayesian approach could be applied. ADT can provide more opportunities to draw quick inferences on the lifetime distribution of highly reliable items under normal use conditions. Meeker and Escobar (1998) studied acceleration models for degradation analysis. Most degradation researches studied the MLE method for model parameters. Lu and Meeker (1993) used a nonlinear mixed-effects degradation model to analyzing degradation data. They used a two-stage approximation method to estimate the parameters and confidence intervals because of not existence of closed forms for MLEs. Bae and Kvam (2004) considered a nonlinear random-coefficients model to study the degradation path of vacuum fluorescent displays. They utilized four different approximation methods for the MLEs and Monte Carlo simulation for deriving failure-time distribution. Analyzing accelerated degradation data by hierarchical modeling approach was respectively discussed by Pan and Crispin (2011) and Park and Bae (2010).

Unlike MLE, the Bayesian approach has not been widely used in degradation analysis. Wakefield et al. (1994) used a Bayesian approach to analyze repeated degradation measures in linear and non-linear models. Wiener process had been utilized for degradation modeling and assumed that failure times follow an Inverse Gaussian distribution by Pettit and Young (1999). They proposed a fully Bayesian method for integrating failure time data with degradation data to derive failure time distributions. A Bayesian linear mixed-effects model used by Onar et al. (2007) to describe the degradation paths for rut depth. For further reading, we refer to Broemeling (2015).

In degradation studies, a critical threshold level of degradation is considered and failure time is based on this level, and distribution of life time data is calculated with estimated degradation model parameters. In this study, we consider degradation as a discrete variable, and measurements are done along time. Sometimes the exact degradation values can not be measured, and we can recognize that the degradation value is greater or less than the finite value. This may be happened because of the lack existence of exact measurement tools, time and cost limitations. Different models can be used to take into account correlations between responses. One possibility is marginal modeling, which can be used to make inferences about parameters averaged over the whole population, for more details, authors refer to Snell (1964). A second possibility is random-effects modeling, which makes inferences about variability between respondents, see Berridge and Dos Santos (1996) and Ware (1985). The basic idea underlying a random-effects model is that there is a natural heterogeneity across individuals in their regression coefficients and that the heterogeneity can be handled by a probability distribution. The third approach would be to use Markov (transition) models which were studied by Anderson and Goodman (1957) and Muenz and Rubinstein (1985). Here, we utilize a generalized linear mixed effect model to demonstrate the relation between discrete responses over time and covariates. Moreover, based on the presented model point estimations based on the Monte-Carlo approximation (MLE) and Bayesian estimations were derived.

The rest of the paper is organized as follows. Section 2, present a degradation random effect model and the likelihood function. In Section 3, a simulation study is presented. Finally, Section 4 concludes this work and provides a brief discussion on future research challenges.

2 Statistical model

Let $D(t),\ t > 0$ be the true degradation path for a unit. In this case, the observed degradation for ith unit at jth measurement time (t_j) could be modeled as a generalized linear mixed effect model as follows:

$$Y_{ij} = D(t_j, \lambda, \beta_i) + \epsilon_{ij}, \qquad i = 1, ..., N,\ j = 1, ..., n_i,$$

where Y_{ij} is observed degradation for ith unit at jth time. The true degradation path is denoted by $D(t_j, \lambda, \beta_i)$, and n_i is number of degradation measurement for unit i, $\lambda = (\lambda_1, ..., \lambda_k)$ is fixed effect vector, $\beta_i = (\beta_{1i}, ..., \beta_{ki})$ is random effect vector for ith unit that makes different paths for each unit , $\beta_i \sim N_k(0, \Sigma)$ and ϵ_{ij} is a measurement error for ith unit at time t_j which has normal distribution(i.e, $\epsilon_{ij} \sim N(0, \sigma_e^2)$). The Elements of β_i vector is independent of ϵ_{ij} which have been mentioned by Meeker and Escobar (1998).

Sometimes the exact amount of degradation can not be measured due to time, cost and, measurement tools constraints. However, we can recognize that it is more or less than a predetermined critical level (threshold). Suppose Y_{ij}, show the increasing degradation value at time t_j, but the exact degradation value is unknown. Define Z_{ij} as follows:

$$Z_{ij} = \begin{cases} 1 & Y_{ij} \geq \frac{D_0}{\alpha_j} \\ 0 & Y_{ij} < \frac{D_0}{\alpha_j}, \end{cases} \tag{1}$$

where, D_0 is the first degradation level and α_j is degradation coefficient of unit i at time t_j. In other word, Z_{ij} is a Bernoulli variable for continuous variable Y_{ij}, that show the state of this continuous variable respect to D_0. In order to analyse the binary degradation values along time, generalized linear mixed effect model is considered as follows:

$$g(P(Z_{ij} = 1|X_{ij}, U_{ij})) = D(t_j; \lambda, \beta_i), \tag{2}$$

where g is an appropriate link function such as logit or probit and D is the true degradation path. Also, λ is contained a vector of fixed effects of explanatory variables, X_{ij}, and β_i are random effects for sub-vector of X_{ij}, U_{ij}. We assume that $\beta_i \sim N_k(0, \Sigma)$, and the response variables are supposed to be independent conditional on random effects.

Let $Z_i = (Z_{i1}, ..., Z_{in_i})$ be a vector of discrete response variable for unit i. In our proposed model, the likelihood function are constructed as follows:

$$\begin{aligned} L(\lambda; Z_i) &= \int_{\beta_{1i}} ... \int_{\beta_{ki}} f(Z_i; \lambda|\beta_i) f(\beta_i) d\beta_{1i}...d\beta_{ki} \\ &= \int_{\beta_{1i}} ... \int_{\beta_{ki}} f(Z_{i1}, ..., Z_{in_i}; \lambda|\beta_i) f(\beta_i) d\beta_{1i}...d\beta_{ki} \\ &= \int_{\beta_{1i}} ... \int_{\beta_{ki}} \prod_{j=1}^{n_i} f(Z_{ij}; \lambda|\beta_i) f(\beta_i) d\beta_{1i}...d\beta_{ki}. \end{aligned}$$

The probability mass function of Z_{ij} is as follows:

$$P(Z_{ij} = z_{ij}; \lambda, \beta_i) = \left(g^{-1}(D(t_j; \lambda, \beta_i))\right)^{z_{ij}} \left(1 - g^{-1}(D(t_j; \lambda, \beta_i))\right)^{1-z_{ij}}.$$

Let $\underline{Z} = (Z_1, \ldots, Z_N)$, then the Likelihood function for N units is given by

$$L(\lambda; \underline{Z}) = \prod_{i=1}^{N} f(Z_i; \lambda) = \prod_{i=1}^{N} \int_{\beta_{1i}} ... \int_{\beta_{ki}} \prod_{j=1}^{n_i} f(Z_{ij}; \lambda|\beta_i) f(\beta_i) d\beta_{1i}...d\beta_{ki}. \tag{3}$$

The MLE of parameters is obtained with maximisation of above function with respect to λ and β_i by Monte Carlo numerical methods.

Bayes estimates could be obtained with considering appropriate priors for parameters. Suppose $\pi(\lambda)$ and $\pi(\Sigma)$ be prior distributions for parameters λ and Σ, respectively. In this case the joint posterior distribution of (λ, Σ) could be written as follows:

$$\begin{aligned} \pi(\lambda, \Sigma|\underline{Z}) &\propto \prod_{i=1}^{N} \left[\int_{\beta_{1i}} ... \int_{\beta_{ki}} \prod_{j=1}^{n_i} f(Z_{ij}|\beta_i) f(\beta_i) d\beta_{1i}...d\beta_{ki} \right] \pi(\lambda)\pi(\Sigma) \\ &\propto \prod_{i=1}^{N} \left[\int_{\beta_{1i}} ... \int_{\beta_{ki}} \prod_{j=1}^{n_i} \left(g^{-1}(D(t_j; \lambda, \beta_i))\right)^{Z_{ij}} \right. \\ &\quad \left. \times \left(1 - g^{-1}(D(t_j; \lambda, \beta_i))\right)^{1-Z_{ij}} f(\beta_i) d\beta_{1i}...d\beta_{ki} \right] \pi(\lambda)\pi(\Sigma). \end{aligned} \tag{4}$$

Then, the conditional posterior distributions of λ and Σ could be calculated as follow

$$\begin{aligned} \pi(\lambda|\Sigma, \underline{Z}) &\propto \int \prod_{i=1}^{N} \left[\int_{\beta_{1i}} ... \int_{\beta_{ki}} \prod_{j=1}^{n_i} \left(g^{-1}(D(t_j; \lambda, \beta_i))\right)^{Z_{ij}} \right. \\ &\quad \left. \times \left(1 - g^{-1}(D(t_j; \lambda, \beta_i))\right)^{1-Z_{ij}} f(\beta_i) d\beta_{1i}...d\beta_{ki} \right] \pi(\lambda)\pi(\Sigma) d\Sigma \\ \pi(\Sigma|\lambda, \underline{Z}) &\propto \int \prod_{i=1}^{N} \left[\int_{\beta_{1i}} ... \int_{\beta_{ki}} \prod_{j=1}^{n_i} \left(g^{-1}(D(t_j; \lambda, \beta_i))\right)^{Z_{ij}} \right. \\ &\quad \left. \times \left(1 - g^{-1}(D(t_j; \lambda, \beta_i))\right)^{1-Z_{ij}} f(\beta_i) d\beta_{1i}...d\beta_{ki} \right] \pi(\lambda)\pi(\Sigma) d\lambda. \end{aligned} \tag{5}$$

Bayesian analysis is used in different and wide fields because of its philosophical consistency, making consistent decisions in uncertain conditions, artful handling of large and complex statistical problems, using powerful computational tools, and providing natural ways to structure data and knowledge to answer questions. Thus we are motivated to use Bayesian analysis in presence of Non-informative, Less-informative, and informative priors distributions using square error loss (SEL) function.

3 Simulation study

First of all, suppose that the expected continuous degradation path is as follow where λ and β_i are fixed and random slope in model, respectively. The expectation of model can be rewritten as:

$$E(Y_{ij}) = (\lambda + \beta_i)T_j. \tag{6}$$

Suppose, there are 100 units ($i = 100$) in 3 times ($j = 3$), in this study. For generating discrete degradation data following steps should have be done.

- Generate random slope $\beta_i \sim N(0, \sigma^2_{\beta_i})$.
- Generate discrete data from Bernoulli distribution $Z_{ij} \sim Bernoulli(\pi_{ij})$ where $\pi_{ij} = P(Y_{ij} \leq (\lambda + \beta_i)T_j)$, $Y_{ij} \sim N(0, 1)$ and $T_j = T, \quad T = 1, 2, 3$.

Table 1: Parameter Estimates and Standard Errors.

		Model1	Model1	Model2
Par	True value	MLE Estimate(SE)	Bayes Estimate(SE)	MLE Estimate (SE)
Priors	Informative Priors			
λ	-1	-1.228(0.591)	-1.156(0.225)	-0.397(0.059)
$\sigma^2_{\beta_i}$	1	1.226(0.833)	1.120(0.271)	-
AIC	-	249	248	305
BIC	-	256	256	309
Priors	Less Informative Priors			
λ	-1	-1.242(0.590)	-1.175(0.459)	-0.396
$\sigma^2_{\beta_i}$	1	1.251(0.830)	1.148(0.640)	-
AIC	-	249	249	305
BIC	-	256	256	309
Priors	Non Informative Priors			
λ	-1	-1.236(0.592)	-1.179(0.477)	-0.395
$\sigma^2_{\beta_i}$	1	1.242(0.835)	1.155(0.680)	-
AIC	-	250	250	306
BIC	-	257	257	309

Simulation was repeated in 1000 iterations and parameters estimate were obtained by considering probit link with using *glmer*, *bglmer* and *glm* in R program. Also appropriate priors for parameters λ and $\sigma^2_{\beta_i}$ are considered Normal and Inverse Gamma distributions, respectively. For estimating parameters, two models are considered. The model 1 considers fixed and random effect vectors concurrently like as assumed model in (6), while random effect vectors were removed in model 2, that is $E(Y_{ij}) = \lambda T_j$. On

Table 2: RRMSE of Parameters.

	Model 1				Model 2
	MLE	Informative	Less-informative	Non-Informative	-
β	0.642	0.237	0.486	0.507	0.608
σ_{b1}^2	0.875	0.272	0.652	0.695	-

the other hand, parameters distribution were considered in three status: Non Informative (IG(0.001,0.001), N(-1,1000)), Less Informative (IG(0.01,0.01), N(-1,100)) and Informative Priors (IG(0.1,0.1), N(-1,10)).

Table 1 show MLEs, Bayesian estimates and standard errors (SE), which were obtained in presence and absence of random effects in model 1 (including both fixed and random effects) and model 1 (including only fixed effects), respectively. As you can see in this table, Model 1 has less AIC and BIC than Model 2 which random effects were been removed.

Table 2 consists relative root mean squared errors of parameters which is denoted by RRMSE ($RRMSE = \frac{\sqrt{(MSE(\hat{\theta}))}}{\theta}$). This table shows RRMSE of Bayes estimates are less than ML estimates. In the Bayesian framework, as we expected, informative Bayes estimates are precise than less informative Bayes estimates and less informative Bayes estimates are precise than non-informative Bayes estimates.

4 Conclusions

Sometimes the exact degradation value could not be measured and only is recognized that is greater or less than a predetermined value, this led us to discrete variables and models. We considered degradation as the discrete response variable and utilized a generalized linear model (mixed-effects model) for modeling repeated measures. As you saw in the previous section, the random effect model is precious than the fixed-effect model and Bayes estimates are accurate than ML estimates. Informative priors resulted in less biasness than less informative priors and less informative priors than non-informative priors.

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